Summary of the Report of the

Advisory Committee on Mathematics to the California State Curriculum Commission

- **The Strands of Mathematics**
- **Mathematics Programs for Teachers**
- **A Study of New Programs and Supplementary Materials**

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BULLETIN OF THE CALIFORNIA
STATE DEPARTMENT OF EDUCATION
SACRAMENTO

VOL. ~~XXXIII~~ 32
No. 6
DECEMBER, 1963

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FOREWORD

The past decade has seen a marked increase in the use of mathematics as a tool for seeking solutions to important problems in scientific fields. Business and industry have adapted mathematical procedures and techniques to promote efficiency in the production and distribution of goods. These and similar developments in other aspects of modern life have caused both mathematicians and educators to question the adequacy of the traditional mathematics curriculum as a means of preparing young people for handling the mathematical concepts and procedures which they will encounter in college and university studies and in their chosen occupations.

In 1960, at the request of the California State Curriculum Commission, the Superintendent of Public Instruction appointed a statewide advisory committee to the Commission to develop recommendations as to needed changes in the mathematics curriculum of the public schools. The State Advisory Committee on Mathematics, which held its first meeting in December of 1960, consisted of 35 persons and included professors of mathematics in California colleges and universities; classroom teachers, both elementary and secondary, and coordinators of curriculum in California public schools; and staff members of California colleges and universities who were responsible for the teacher education programs the schools were offering. E. G. Begle, Professor of Mathematics at Yale University, was appointed consultant to the Committee.

The State Advisory Committee on Mathematics presented its report to the Curriculum Commission in two parts on March 8 and 14, 1962. The Commission requested that the substance of this report be printed so that it could be circulated to school people throughout the state. On June 14, 1962, a preliminary report on the work of the State Advisory Committee was presented to the State Board of Education; the Board approved the printing of the report.

The State Curriculum Commission, the State Board of Education, and the State Department of Education appreciate the willingness of the members of the State Advisory Committee to undertake the task assigned them. I am confident that the investment of time and energy that has been required to complete this project will pay dividends in terms of a substantial improvement in the mathematics curriculum of the public schools.

Superintendent of Public Instruction

PREFACE

At the first meeting of the State Advisory Committee on Mathematics, held in December, 1960, Kenneth P. Bailey, a member of the State Curriculum Commission, set forth the objectives which the Commission believed should guide the Committee's work. Briefly stated these objectives were (1) to evaluate the several current proposals for the improvement of the mathematics curriculum; (2) to develop recommendations as to needed improvements in the California mathematics program; (3) to promote the development of improved instructional materials; and (4) to make recommendations concerning needed improvements in teacher education in the field of mathematics.

With these objectives in mind and in recognition of the fact that the Committee had only some 16 months in which to complete its assignment, the Committee divided into three principal working subcommittees. The first was a subcommittee on "Strands of Mathematical Ideas." The task assigned to this group was that of identifying and describing the essential integrating strands of mathematics that should be incorporated into the program for kindergarten and grades one through eight.

A second subcommittee was appointed to investigate the preservice education in mathematics of elementary school teachers and to study methods and procedures for organizing and conducting inservice education programs for teachers at both the elementary and secondary levels.

The third subcommittee was assigned the job of investigating the more recent "new" mathematics programs that have attracted nationwide attention and studying the commercially produced materials that could be used profitably to supplement the state adopted textbooks. This group was requested to report immediately upon materials that would beneficially supplement instructional materials currently adopted for the elementary grades.

The substance of the material in this publication comes from the efforts of the Advisory Committee on Mathematics and its subcommittees. The results of the investigations conducted by these committees and recommendations made by them should exert a stimulating and helpful influence upon mathematics programs in California schools for many years to come.

RICHARD M. CLOWES
Associate Superintendent of Public Instruction; and Chief, Division of Instruction

THE STATE ADVISORY COMMITTEE ON MATHEMATICS

The names of the members of the State Advisory Committee on Mathematics and the positions they held at the time of the study follow:

L. J. Adams, Chairman, Department of Mathematics, Santa Monica City College, Santa Monica Unified School District

G. Don Alkire, Professor of Mathematics, Fresno State College

Harold Allison, Principal, Sir Francis Drake High School, Tamalpais Union High School District, San Anselmo

Emily V. Baker, Curriculum Consultant and Coordinator of Elementary Arithmetic and Geography, office of San Bernardino County Superintendent of Schools

Leslie S. Beatty, General Supervisor of Elementary Education, Chula Vista Elementary School District

Clifford Bell, Professor of Mathematics, University of California, Los Angeles

Clyde L. Corcoran, Chairman, Department of Mathematics, California High School, Whittier Union High School District

Dan T. Dawson, Executive Secretary, California Elementary Administrators Association, Burlingame

Richard A. Dean, Associate Professor of Mathematics, California Institute of Technology, Pasadena

Lovelle C. Downing, Director of Curriculum and Assistant Director of Inservice Training, Modesto City Elementary and High School Districts

Roy Dubisch, Previous Chairman, Department of Mathematics, Fresno State College (Resigned to accept a position at the University of Washington after serving on the committee during 1960-61)

Lawrence D. Hawkinson, Coordinator of Mathematics, George Washington High School, San Francisco City Unified School District

James Hood, Teacher, San Jose Senior High School, San Jose Unified School District

Earl J. Hoppes, Coordinator of Mathematics and Science, Twentynine Palms High School, Morongo Unified School District

John L. Kelley, Professor of Mathematics, University of California, Berkeley

L. Clark Lay, Professor of Mathematics and Education, Orange County State College, Fullerton

Richard Madden, Professor of Education, Sonoma State College, Cotati

Marvin R. Matthews, Teacher, Chula Vista Junior High School, Sweetwater Union High School District, San Diego County

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William G. Mehl, Teacher, Wilson Junior High School, Pasadena Unified School District (Replacement for Owen J. Parry during 1961-62)

Frances Mettler, Teacher, Walter Hays Elementary School, Palo Alto Unified School District

H. Stewart Moredock, Chairman, Division of Science and Mathematics, Sacramento State College

Owen J. Parry, Teacher, Eugene Field Elementary School, Pasadena City Elementary and High School districts (Resigned to become an exchange teacher in England after serving on the committee during 1960-1961)

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C. Richard Purdy, Dean of Instruction, Alameda County State College, Hayward

Arden Ruddell, Associate Professor of Education, University of California, Berkeley (Replacement for Roy Dubisch during 1961-62)

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Ruth Stone, Chairman, Department of Mathematics, Lewis M. Terman Junior High School, Palo Alto Unified School District

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Jack R. Walker, Teacher, Strathern Street Elementary School, Los Angeles City Unified School District

Thomas F. Wauchope, Teacher, Mt. Whitney High School, Visalia Union High School District

Edward H. Whitmore, Associate Professor of Mathematics, San Francisco State College

Burrell E. Wilson, Chairman, Department of Mathematics, Redwood Junior High School, Napa Union High School District

The names of the ex-officio members of the State Advisory Committee on Mathematics and the positions they held at the time of the study follow:

Kenneth P. Bailey, Principal, Pacific High School, San Bernardino; representative of the State Curriculum Commission

E. G. Begle, Professor of Mathematics Education, Stanford University; Chief Consultant to the Advisory Committee on Mathematics

Gerald D. Cresci, Consultant in Junior College Education, California State Department of Education

Helen Heffernan, Chief, Bureau of Elementary Education, California State Department of Education

Donald E. Kitch, Chief, Supplemental Education Services, California State Department of Education; Chairman of the Advisory Committee on Mathematics

Frank B. Lindsay, Chief, Bureau of Secondary Education, California State Department of Education

George H. McMeen, Special Consultant in Mathematics, California State Department of Education (on leave of absence from California State Polytechnic College during 1961-62)

George L. Roehr, Consultant in Secondary Education, California State Department of Education (on leave of absence to the School Mathematics Study Group, Stanford University, during 1961-62)

Robert S. Shute, Consultant in Elementary Education, California State Department of Education

Wesley P. Smith, State Director of Vocational Education, California State Department of Education

SUBCOMMITTEES

The names of the members of each subcommittee and the position they held at the time of the study follow:

STRANDS OF MATHEMATICAL IDEAS

Richard C. Purdy, *Chairman*

L. J. Adams	Roy Dubisch	Jesse Peckenham
G. Don Alkire	Lawrence D. Hawkinson	Philip Schneider
Emily V. Baker	James Hood	William E. Simmons
E. G. Begle	Richard Madden	Kenneth C. Skeen
Gerald D. Cresci	Marvin R. Matthews	Ruth Stone
Richard A. Dean	H. Stewart Moredock	Jack R. Walker

IMPLEMENTATION: TEACHER AND PILOT PROGRAMS

William F. McClintock, *Chairman*

Harold Allison	Clark L. Lay	Thomas F. Wauchope
Clyde L. Corcoran	Mary E. McDermott	Edward H. Whitmore
	Robert S. Shute	

NEW PROGRAMS AND MATERIALS

Leslie S. Beatty, *Chairman*

Clifford Bell	William G. Mehl	Thomas H. Southard
Lovelle C. Downing	Frances Mettler	George R. Vojtko
Earl J. Hoppes	Owen J. Parry	Burrell E. Wilson
	Arden Ruddell	

CALIFORNIA STATE CURRICULUM COMMISSION

The names of the members of the California State Curriculum Commission at the time the study was made and their respective positions follow:

Irwin O. Addicott, Vice-President, Fresno State College

Kenneth P. Bailey, Principal, Pacific High School, San Bernardino

Laura Henry Doyle, Head, Mathematics Department, Menlo-Atherton High School, Atherton

Fernie M. Gleason, Teacher, Alvin Avenue Elementary School, Santa Maria

* Mrs. Gertrude R. Goodrich, Principal, Theodore Judah Elementary School, Sacramento

* Tennessee Kent, Assistant Superintendent in Charge of Elementary Schools, San Francisco Unified School District

Samuel C. McCulloch, Dean of the College, San Francisco State College

** Mrs. Martha K. McIntosh, Assistant Superintendent in Charge of Elementary Schools, San Diego City Unified School District

H. M. McPherson, Superintendent, Napa School Districts

Jackson Price, District Superintendent, Shasta Union High School District and Junior College, Redding

Mrs. Jeannine R. Sevey, Vice-Principal, South Oceanside Elementary School, Oceanside

C. C. Trillingham, Superintendent of Schools, Los Angeles County

Mrs. LaVon H. Whitehouse, Assistant to Director of Curriculum and Elementary Curriculum Coordinator, Division of Instructional Services, Los Angeles Unified School District

Roy E. Simpson, Superintendent of Public Instruction and Director of Education, Chairman of the Curriculum Commission, Sacramento

Richard M. Clowes, Associate Superintendent of Public Instruction; and Chief, Division of Instruction; Secretary to the Curriculum Commission, Sacramento

* Term expired August 29, 1961.
** Resigned effective January 15, 1962.

CONTENTS

PART ONE

THE STRANDS OF MATHEMATICS

Cultural changes occurring in our time have significant implications for the curriculum as a whole and for mathematics in particular. The technological revolution which is now taking place makes it imperative that every citizen have some understanding of mathematical reasoning and that, at every level of proficiency, a much larger group have an understanding of mathematical method. In addition to the traditional uses of mathematics in commerce and the physical sciences, we now have mathematical models in the behavioral sciences, computers of a speed and complexity to challenge the most creative programmer, new problems in decision making, probabilistic mathematics in all areas from business to medical research to quantum mechanics, and a vast host of new applications of mathematics both old and new. The ability to grasp and analyze complex situations is needed by an ever increasing proportion of the population. Every child should at least be given the opportunity to learn the mathematics which will make these activities accessible to him.

At the same time the traditional uses of mathematics are still highly important, and we need a vast number of citizens proficient in the applications of mathematics to every day activity in the home, business, industry, and government. We need a mathematics program for kindergarten and grades one through eight which will contribute to more efficient learning of these specialized skills and which will at the same time make easier the learning of new skills which the future promises to require.

Our task is to provide every pupil today with the mathematical instruction which will be most useful to him tomorrow and which will make him most useful to society. This task must be accomplished in the most efficient manner possible, for mathematics is but one of the areas of the curriculum which is faced with increasing demands. To achieve our purpose, we must draw upon the best that is known in the field of mathematics and in the fields of learning and teaching.

The curriculum which we recommend departs but little from the topics normally studied in kindergarten and grades one through eight, topics which long ago proved their enduring usefulness. But it is essential that this curriculum be presented as one indivisible whole in which the many skills and techniques which compose the present curriculum are tied together by a few basic strands of fundamental concepts which run through the entire curriculum. These strands are described individually in the body of this report.[1]

We envisage a mathematical curriculum which is designed for pupils of all ability levels, since all pupils will live in the same technological culture. Those mathematical concepts which underlie the K-8 studies are important to the machinist as well as to the stargazer. Each unit in the mathematical presentation must contribute to the understanding and insight which is our objective. It is to be expected, however, that different pupils will attain different levels of understanding and skill. The curriculum should provide materials and experiences which take into account these different levels.

Definite grade placement of topics is not crucial. We must recognize that concepts are not mastered immediately but are built up over a period of time. It is therefore important that these notions be introduced as early as possible and be linked with the pupils' mode of thinking. Above all we seek, to the limit of each pupil's capability, his understanding of that unified mathematical structure which is the content of K-8 mathematical study.

The basic strands in the K-8 curriculum are *numbers and operations, geometry, measurement,* and *applications of mathematics.* But these are not enough. There have been remarkable advances in this century in the clarity and precision of mathematical discourse. We have come to see that many apparently diverse notions are special cases of a few underlying concepts. We believe it essential that mathematical instruction bring some of this economy of unified thought to the pupil. Consequently, we recommend a planned and conscious development of certain strands which contribute to this clarity, precision, and economy. These are the strands that are labeled *sets, functions, logic,* and *the mathematical sentence.*

Finally, we recognize and emphasize that a sound curriculum is not sufficient. Sound and meaningful concepts can be just as poorly taught as unsound concepts. A reorientation of the mathematical program which emphasizes structural aspects will be unsuccessful unless the pedagogy is successful.

Good mathematical instruction has a dynamic character. Pupils should be encouraged to make conjectures and guesses, to experiment

[1] Since the current curriculum is well understood, more space is given in this report to the newer concepts. The space devoted here to a concept, therefore, is not to be construed as a measure of its importance.

and to formulate hypotheses, and to seek meaning. The applications of mathematics provide a wealth of possibilities for this kind of mathematical activity. And for many pupils such involvement is an important source of motivation. Practice materials should be planned to elicit thoughtful response, to bring out relationships, to develop understanding, and to help the pupil see his progress.

It is important that textbooks exhibit this dynamic approach to mathematics. Mathematical concepts are built up through a sequence of instances, not through a list of procedural rules. Mathematical abstraction can be formulated by pupils as well as by professional mathematicians.

STRAND 1

Numbers and Operations

The K-8 program of mathematics instruction in the area of numbers and operations has three basic objectives, namely:

- A fundamental understanding of the system of rational numbers, with some preparation and resultant readiness for the notion of real numbers
- Computational skill which encompasses (a) facility with the number facts and (b) understanding of the principles of operations and of the positional system, with particular insight into computational algorithms
- Ability to apply knowledge of numbers in relatively simple but increasingly complex situations

These three objectives must be kept in balance and allowed to interact throughout the K-8 program. If this section seems to dwell mostly upon the first of these objectives, it is because of the recent widespread concern that this objective be raised to a level of emphasis equal to that of the other two objectives. Some examples are provided to show how an understanding of principles can relate to computation. These examples are *not* to be construed as advocating a single approach to the study of any topic. There must be flexibility which can be attained only through many approaches; space does not permit adequate treatment of these important pedagogical matters. Applications are a source of interest and motivation for pupils and can aid in clarifying mathematical ideas. This important subject is discussed separately in greater detail in a subsequent section.

Understanding the Rational Number System

In mathematics, the expression "number system" has the technical meaning of a logical system or a postulational oragnization of ideas about numbers consisting of principles (axioms), definitions, and related concepts (theorems). The rational number system is studied in the elementary school program by means of certain subsystems. First encountered is the system of natural (counting) numbers and zero [0, 1, 2, 3, . . .], sometimes referred to as whole numbers. These numbers are ordered, named in positional notation, and the basic operations with their properties are introduced. The system of fractions (positive rational numbers and zero), sometimes referred to as fractional numbers, is studied next. Such numbers as 0, ½, ⅔, $\frac{5}{4}$, 1⅜, 2, and all the numbers commonly referred to as common, proper, and improper fractions; the natural numbers and zero; and mixed numbers are included in the system of rational numbers. The operations and principles developed for the natural numbers are extended to fractions. The negative rational numbers are introduced by first considering the numbers —1, —2, —3, . . ., and then later such numbers as —¼, —⅞, and —1⅔. The operations and principles are extended to include these numbers.

Preparatory activities for the study of the real numbers includes the decimal expansion of numbers and showing that the $\sqrt{2}$ is not a rational number (and is therefore called an irrational number). The decimal expansion of a rational number is periodic (e.g., .7143143 . . ., .5333 . . ., or .25000 . . .) and conversely a periodic decimal names a rational number. As a supplementary activity in the later grades, some single mathematical system such as a finite arithmetic can be studied by some pupils to deepen their understanding of the nature of a mathematical system.

Certain key principles and concepts play an important part throughout the study of the rational number system, and in a very real sense they unify the whole development. The arithmetic of the K-8 curriculum is a single monolithic mathematical structure, and it must be taught in a sequential, developmental fashion. It must *not* appear to the pupil as a sequence of disconnected fragments or computational tricks. Some of the important unifying ideas are discussed briefly in the following section of this report.

Unifying Principles for the Rational Number System

EARLY DEVELOPMENTAL STAGES, ONE-TO-ONE CORRESPONDENCE

The early number program should provide many opportunities for pupils to differentiate relative sizes from among sets of two, three, and

four objects without resorting to counting. When four blocks are taken from a pupil's view and only three returned, he is likely to perceive that a smaller number have been returned even though he is unable to count in terms of the number system. Through experiences of this kind he learns the language of comparison in number and measure such as *more, many, fewer, bigger, longer, farther, thicker, faster,* and the like. At this stage the concept of *more* has not yet been differentiated as between magnitude and multitude. Such experiences help form the needed background for emerging patterns of numeration and notation.

The pupil has had many preschool experiences with matching things one-to-one, two-to-two, and so on, and he has been contrasting inequalities of sets of one, two, three, and more. The next step in his learning is to associate the number names and symbols with sets of objects. The study of number relationships is promoted by arranging sets of objects (e.g., blocks) in sequential order, each set being one more than the preceding set. Experiences of this type help the pupil see the order behind our system of numeration and to associate the number words and symbols more easily with sets of objects. Because of the relative complexity of learning our sets of arbitrary counting words and numerals, a great deal of practice in a variety of situations is necessary if confusion resulting from partial learnings is to be avoided.

After counting with understanding has been achieved with both cardinal and ordinal numbers, pupils should learn how to write numerals. Writing beyond the numeral 9 should be accompanied with the concept of place value. As they count past nine, pupils should either be taught or led to sense the principle of repetition in our system of numeration even though they do not fully comprehend place value at the time. As they learn to write numbers beyond nine, the less refined notions of repetition must be clearly organized into concepts of place value.

PLACE VALUE

Place value is the fundamental principle of organizing enumeration beyond nine. In the decimal system of enumeration, the ten digits—0, 1, 2, 3, 4, 5, 6, 7, 8, 9—can be placed in an infinite number of positions arranged according to ordered powers of base ten, such as hundredths, tenths, units, tens, hundreds, . . ., to express any real number.

A child's concept of place value may have its beginning in preschool or kindergarten number experiences. The concept is refined and extended in the primary number program. From discovering that bundles of ten are counted in a similar fashion to single objects, a pupil should quickly move to seeing one symbolic object in tens' place as being equivalent to ten of them in ones' place. He learns that the number 14 can be represented by 1 object in tens' place and 4 objects in ones' place, as on an abacus.

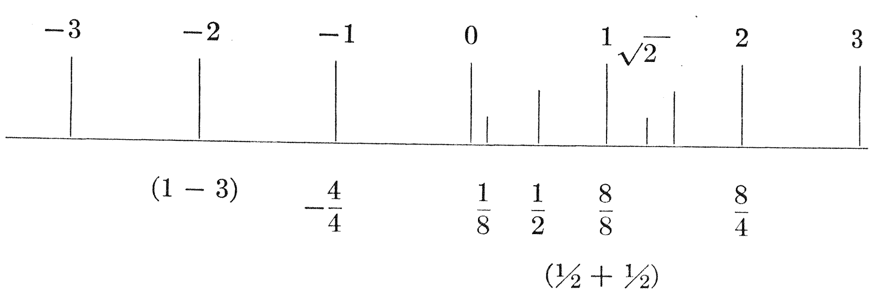
A succession of experiences to extend the concept follow rapidly, as for example writing numerals for large numbers, carrying in addition and in multiplication, and regrouping in subtraction. To aid in understanding the concept of place value, pupils should be taught to work with bases other than ten. Once generalized, the concept of place value is of invaluable service in such situations as computation with decimal fractions, percents, rounding in approximate computation, exponents, logarithms, and scientific notation.

NUMBER AND NUMERAL

Pupils should mature as rapidly as possible in conceiving of number as an abstraction and of a numeral as its symbol. A number is a mathematical concept; a numeral is a written symbol for a number and is the name of a number. The number 8 is the concept which the pupil associates with any set which has the property of containing eight members. Numerals are the first of the mathematical symbols which a pupil learns to use in the elementary school mathematics program. Symbols such as $+$, $=$, $\neq$, $<$, $>$, and $\times$ come later.

ORDER: THE NUMBER LINE

The ordering of the natural numbers should be emphasized from the beginning. The use of the number line makes vivid many properties of numbers. The number line and segments of the number line should be introduced in preparation for later work. A sketch, such as the one below, exhibits many important facts. By means of it, one can visualize many names for the same number.



There is no smallest fractional number, and there is no largest whole number. Numbers increase as we go to the right, and "less than" means simply "to the left of." Thus any negative number is to the left of 0 and is hence less than 0, and -4 is less than -1. Furthermore, this association of numbers with points on the number line serves as essential preparedness for the later study of the coordinate plane.

OPERATIONS

Much of arithmetic is concerned with the four fundamental operations. These are defined first for the whole numbers. Certain fundamental principles are emphasized which help to clarify the extension of the operations to larger systems of numbers. Addition is initially defined in terms of counting (technically, in terms of counting the members of the union of two disjoint sets). Addition is an operation which assigns to a pair of numbers another number called the sum. Subtraction, in turn, may be defined in terms of addition; it is the operation for finding one addend if the sum and the other addend are known. The number $7-3$ is that number which, when added to 3, yields 7. If this is clearly understood, many of the subtraction facts can be learned simultaneously with the addition facts. The "take away" interpretation of subtraction should not be used to an extent which obscures the fact that subtraction is the inverse operation of addition.

The concept of multiplication presented in the elementary school up to the present has usually been limited to that of repeated addition. While it is not suggested that this approach be abandoned, we do recommend that it be supplemented by the cartesian-product interpretation. This interpretation, of which most elementary teachers have little or no knowledge at present, has such important advantages as (1) application to types of problem situations which cannot be adequately treated through repeated addition and (2) creation of a geometrical model (array) which facilitates analysis of the product.

Through the cartesian-product interpretation of multiplication, we pair the elements of one set with the elements of another set and then enumerate how many pairs result. Suppose for example we encounter the need to multiply 3×4, arising perhaps from some social or physical situation. Overlooking any labels which may be attached to these numbers, we can picture the elements of each set with any abstract symbols we wish and then pair them. (The lines in Figure 1 illustrate the possible pairings.) Each of the pairings can then be represented more simply with some new abstract symbol, thus creating an array. (See Figure 2.) We can then determine the number of elements in the array by a variety of methods, repeated addition being one good method.

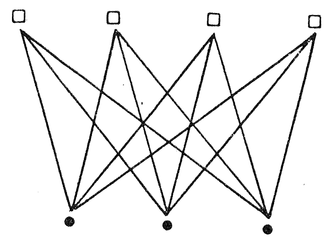


Figure 1

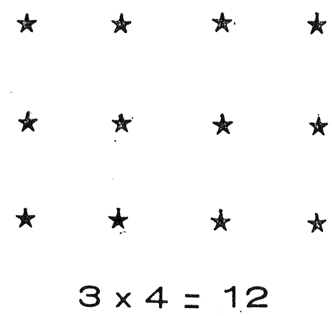


Figure 2

Division should be defined in terms of multiplication, just as subtraction is defined in terms of addition. As pupils learn that $3 \times 2 = 6$, they should also learn that $6 \div 2$ is the number which multiplies 2 to produce 6. Later they may learn that the number ¾ is by definition that number which when multiplied by 4 produces 3, resulting in the equality $4 \cdot \frac{3}{4} = 3$. Physical interpretations and applications should be used to emphasize and develop such arithmetical generalizations. The division facts can in large part be learned simultaneously with the corresponding multiplication facts. For example, these number facts should be learned together: $6 \times 2 = 12$, $12 \div 2 = 6$, $2 \times 6 = 12$, and $12 \div 6 = 2$. The intimate relationship between multiplication and division must be emphasized.

CLOSURE

There are several principles which unify the development of the different number systems. Addition, subtraction, multiplication, and division have in common the fact that they are binary operations in the sense that operation with certain pairs of numbers results in assigning a number.

Since the sum or product of any two whole numbers—0, 1, 2, 3, . . . —is a whole number, whole numbers are said to be "closed" under the operations of addition and multiplication. On the other hand the whole numbers are not closed under the operations of subtraction and division; for example, the numbers resulting from $3 - 6$, $2 \div 5$, are not whole numbers.

The rational numbers are closed (have closure) under all four operations, with the single exception of division by zero. (3/0 does not exist, since there is no number which when multiplied by 0 produces 3.) The foregoing principles of closure should have been taught by the end of the K-8 mathematics curriculum.

COMMUTATIVITY

Both addition and multiplication are commutative; that is, the order in which two whole numbers are added or multiplied does not affect the sum or the product. Thus, $2 + 3 = 3 + 2 = 5$ and $2 \times 3 = 3 \times 2 = 6$. By counting sets and arrays, pupils readily discover that addition and multiplication of whole numbers are commutative. This principle has been used very widely by teachers in what is known as teaching by "families." It is less obvious, perhaps, that ¼ of 3 is equal to ¾ of 1, although convincing physical illustrations can and should be given. In any case, commutativity should be emphasized, and the pupil should understand, by contrast, that subtraction and division are not commutative. Thus $2 - 3 \neq 3 - 2$ and $2 \div 3 \neq 3 \div 2$.

The teacher should know and understand mathematical terms such as commutativity, associativity, function, and so on. However, such terminology should not be used prematurely with pupils.[2] One would expect that by the end of the elementary school mathematics program, the pupil would be familiar with the standard mathematical terminology for simple arithmetic concepts.

ASSOCIATIVITY

Both addition and multiplication are associative; that is, the way in which the numbers are grouped for addition or for multiplication does not affect the sum or product. Thus $(2 + 7) + 6 = 9 + 6 = 15$; and $2 + (7 + 6) = 2 + 13 = 15$; and $(2 \times 7) \times 6 = 14 \times 6 = 84$; and $2 \times (7 \times 6) = 2 \times 42 = 84$. By contrast, neither subtraction nor division is associative. From an example illustrating subtraction we see that $(7 - 4) - 2 = 3 - 2 = 1$, but $7 - (4 - 2) = 7 - 2 = 5$. The associativity of addition and multiplication of whole numbers is easily justified intuitively and is widely useful, but the importance of the concept goes beyond whole numbers. As an example, let us see how the commutative and associative properties can be applied to a problem involving fractional numbers where the situation is intuitively

[2] With regard to the appropriateness of introducing new terminology, a sound working principle might be that at least two different instances of a concept should be encountered before it is named.

more difficult. Suppose that we are asked to find $\frac{2}{3} \times \frac{5}{7}$ and that we already understand that $\frac{2}{3} = 2(\frac{1}{3})$ and $\frac{1}{3} \times \frac{1}{7} = \frac{1 \times 1}{3 \times 7}$. We may then apply the communtative and associative principles as follows:

$$\tfrac{2}{3} \times \tfrac{5}{7} = 2(\tfrac{1}{3}) \times 5(\tfrac{1}{7}) = 2 \times 5 \times (\tfrac{1}{3}) \times (\tfrac{1}{7}) = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}.$$

Of course, in teaching the computation of products of fractional numbers, one would hope that many illustrations and applications would be used to fix the procedures in the pupil's mind. Nevertheless, the underlying mathematical principle, whose use we have just illustrated, should not be neglected.

IDENTITY ELEMENTS—ZERO AND ONE

The numbers 0 and 1 have special properties with respect to addition and multiplication: The addition of 0 to a number results in the same number, and so does multiplication of a number by 1. Thus $8 + 0 = 8 = 0 + 8$ and $9 \times 1 = 9 = 1 \times 9$. We say that 0 is the *identity element* with respect to addition and that 1 is the *identity element* with respect to multiplication. These properties of 0 and 1 with respect to the addition and multiplication of whole numbers appear transparently simple. The major importance of the properties, and the reason they must be emphasized in the teaching of arithmetic, is their role in the more complicated study of fractional and rational numbers. The following two examples illustrate the importance of 1 as an identity element in multiplication when expressed in appropriate forms (e.g., $\frac{3 \times 7}{3 \times 7}$, $\frac{4}{4}$, and $\frac{3}{3}$). The first case is for division of fractions.

$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3}}{\frac{5}{7}} \times 1 = \frac{\frac{2}{3}}{\frac{5}{7}} \times \frac{3 \times 7}{3 \times 7} = \frac{\left(\frac{2}{3} \times 3\right) \times 7}{\left(\frac{5}{7} \times 7\right) \times 3} = \frac{2 \times 7}{5 \times 3} = \frac{14}{15}$$

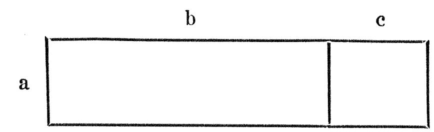
As a second example, we consider the problem of deciding between $\frac{3}{4}$ and $\frac{2}{3}$ as to which is the larger. If we recognize that $\frac{3}{4} = \frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$, and that $\frac{2}{3} = \frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$, the problem is easy.

DISTRIBUTIVITY

The distributivity of multiplication over addition is essential for arithmetic and fundamental to teaching of factoring of polynomials in

the ninth grade. The principle of distributivity states that the product of a number and the sum of two numbers is the sum of the products of the first number and each of the addends. This principle clarifies the solution of the following simple problem: "If a man earns $3\frac{1}{2}$ dollars per hour, how many dollars will he earn in 8 hours?" Employing the distributive principle, $8 \times 3\frac{1}{2} = 8(3 + \frac{1}{2}) = 8(3) + 8(\frac{1}{2}) = 24 + 4 = 28$. In decimal notation the principle is employed in exactly the same way, $8 \times 3.50 = 8(3 + .50) = 8(3) + 8(.50) = 24 + 4 = 28$. Pupils should understand that all of the algorithms for multiplying whole numbers depend upon the distributive principle. For example, $3 \times 13 = 3(10 + 3) = 3(10) + 3(3) = 39$. Stated algebraically and interpreted geometrically the principle becomes:

$$a(b + c) = a(b) + a(c)$$



The distributive principle also plays a fundamental role in computing the sum of two fractions having the same denominator. Thus

$$\frac{2}{3} + \frac{5}{3} = \frac{1}{3}(2 + 5) = \frac{1}{3}(7) = \frac{7}{3}.$$

Knowledge of the foregoing principles upon which a number system must be built is essential for a sound mathematical foundation. They are as important to understanding mathematics as are the concepts of numeration, although the latter have been given much more attention in the elementary school curriculum. Although systems of numeration have been taught to pupils quite generally, there are additional experiences of learning about numeration that deepen pupils' knowledge and also serve to evaluate the quality of pupils' generalization.

Systems of Numeration

Our system of numeration, the decimal system, is based on powers of 10. Thus 2783 is our shorthand notation for $2(10)^3 + 7(10)^2 + 8(10)^1 + 3(10)^0$. An understanding of the nature of this system and of the computational algorithms which make use of the system is probably best obtained by some computation in a different system of numeration. The numeration system with base five, for example, would involve only the digits 0, 1, 2, 3, and 4. The number which in this system is denoted 231 would in expanded decimal notation be $2(5)^2 + 3(5)^1 + 1(5)^0 = 50 + 15 + 1 = 66$. We do not advocate any large amount of computation using different systems of numeration, nor do we insist that our students should, like certain computing machines, be expert in computation in the numeration system of base two. But some study using a base other than ten is recommended in order to emphasize the important roles of *base, place,* and *principles of operation* in our decimal system.

The system of Roman numerals can be used effectively to show its inadequacy as a system of numeration. Although the Romans computed with the abacus, their complicated system of numeration was a handicap to the development of algorithms.

Bases of five, two, eight, and twelve can be taught in the elementary school program so that they can be understood by the pupils. The purpose is not primarily to teach pupils to compute in nondecimal bases but rather to improve their understanding of the principles of base, place value, and operation in our decimal system.

In the Decimal System

FACTORS, MULTIPLES, PRIMES

Computation with fractions requires some knowledge of factors and multiples, and many opportunities for enrichment of the program can result from a study of some of the properties of these numbers. If one considers the set of natural numbers which are multiples of 4 and likewise the set of natural numbers which are multiples of 6, the sets are $\{4, 8, 12, 16, \ldots\}$ and $\{6, 12, 18, \ldots\}$. The set of common multiples is $\{12, 24, \ldots\}$ which is the intersection of the first two sets.

Pupils may profitably be taught the tests for divisibility of numbers in order to determine whether a given number is prime or composite. Knowledge of these tests can be helpful in such situations as determining whether a fraction is expressed in lowest terms.

ATTAINMENT OF COMPUTATIONAL SKILLS

Our second major objective, facility in computation, cannot be separated from the significant contribution made to it by an understanding of the principles already emphasized. Space does not permit an elaboration of the content or pedagogic principles of teaching such essentials as the fundamental facts of addition, subtraction, multiplication, and division. We have already discussed the application of the principles to the algorithms.

Perhaps it is sufficient here to say that we urge achievement of a degree of mastery of facts and algorithms which enables pupils to think through mathematical situations without being cluttered mentally by errors. A reasonable amount of practice is required to attain this goal. Practice should elicit thoughtful responses, establish relationships, and develop understanding. The amounts and types of practice should be adjusted to the individual needs of pupils.

It is doubtful that either sound knowledge of principles or mastery of computation can stand well alone or need to stand alone. One enhances the other not only in its usefulness but also in its attainment. We do not believe that the burden of learning the basic content is so great for any class of pupils as to exclude the study of the principles just described. Knowledge gained in the absence of understanding is soon forgotten and not readily transferable to different siutations.

SQUARE ROOT

The "estimate and average" method of teaching square root is recommended. This works as follows. Suppose we wish to compute the square root of 29. Since $5^2 = 25 < 29 < 36 = 6^2$, we might guess that 5 is approximately the square root. If we divide 29 by its square root, the quotient would be again the square root. Dividing 29 by 5 we obtain 5.8, and we know that the square root lies between 5 and 5.8. Our second estimate is the average, 5.4, of 5 and 5.8. The quotient of $29 \div 5.4$ is 5.370, leading to a next estimate of the average, $\frac{1}{2}(5.4 + 5.370) = 5.385$. The process may be continued in this fashion until the desired degree of accuracy is obtained as evidenced by agreement between divisor and quotient to the desired number of places. This procedure is particularly well adapted to machine calculation.

The method of computing square roots taught in the past and still taught by some teachers should be deferred, if taught at all, until a later algebra course.

STRAND 2

Geometry

The committee recommends that more attention be given to early and progressive development of geometrical concepts in the elementary grades. A conceptual development of some of the important geometrical ideas appropriate for the elementary grades is presented briefly in the following paragraphs.

The introductory stage consists of becoming acquainted informally with some standard geometric shapes and forms. These would include such plane shapes as triangles, rectangles, squares, circles, and ellipses;

and such solid forms as spheres, cubes, rectangular prisms (boxes), cones, and cylinders. Familiarity with these shapes and forms is gained through pictures, models, and examples from everyday life. Some properties of these geometric figures may be informally developed so that pupils can identify the figures.

Following this recognition and identification level, the analytical stage begins. Some basic nonmetric concepts concerning points, lines, and planes are necessary for analyzing properties of geometric figures. A few of these concepts are presented in an intuitive setting: A point is conceived as a fixed location or position, having no size and represented by a small dot. A line (straight line) may be represented by a tightly stretched string or a tracing of a pencil along a straightedge. A line extends indefinitely in both directions. A line is a set of points and is uniquely identified by any two of its points. A line segment is a subset of a line and is formed by two endpoints and all points in between. A ray is a subset of a line having a starting point and all points on one side of the starting point.

A plane is conceived as a flat surface that extends outward indefinitely and which contains many figures (sets of points). A simple closed figure can be traced by starting at a point and coming back to it without crossing the path traced. Polygons are simple closed figures that consist of line segments. An angle consists of two rays which have a common vertex. Finger tracing of models of various solids can be used to develop concepts of surface (flat and curved), face, edge, and vertex. Faces intersect in edges, and vertices are the intersections of edges.

A basic concept in geometry is that of congruence. Two geometrical figures are said to be congruent if they have the same size and shape so that one of them can be superimposed exactly upon the other. Through exploratory activities pupils can be led to discover that line segments with equal measures are congruent, and likewise angles with equal measures are congruent. Minimum requirements for a pair of triangles to be congruent can be identified and the relationships of their measures studied. For two parallel lines and a transversal, the measures of alternate interior, alternate exterior, and corresponding angles that are formed can be studied. With these concepts, the properties of parallelograms can be developed. Angle sums of various polygons can be examined. For circles, the measure relationships of radii, diameters, circumferences, central angles, inscribed angles, and intercepted arcs can be analyzed. Location grids for planes and spheres may be analyzed and related to maps.

The basic geometric constructions using straightedge, compass, protractor, and other instruments should be developed and applied to constructing some geometric figures with certain conditions stipulated. Some scale and perspective drawing using ruler and protractor should be introduced at this time, and the applications of geometry should be developed further.

This is the basic content of informal geometry which should be completed by most pupils by the end of the eighth grade. It provides the necessary background for a more formal study of geometry later or for the more specialized trade and vocational study which uses these geometric ideas.

One comment is necessary, however, concerning the sequential development of this strand. It is important that every opportunity be employed to connect geometry with arithmetic. The use of an array in discussing multiplication of numbers, for example, provides a geometric illustration of an arithmetic operation. More important, early and continued use of the number line in arithmetic provides the proper intuitive foundation for the eventual identification of the Euclidean line with the set of real numbers, the starting point of analytic geometry. Indeed, it is to be hoped that by the end of the eighth grade, a substantial number of pupils will have been introduced also to the number plane.

STRAND 3

MEASUREMENT

Measurement is an integral part of everyday life, so much so that we are often unaware of its extensive use and involvement in our thoughts, observations, and making of decisions. Measurement is a key process in the applications of mathematics since it is a connecting link between mathematics and our physical and social environment. For these reasons it is vitally important that all pupils engage in an analytical study of the measuring process. Since the process involves numbers, a large part of its study should be included in the elementary school arithmetic curriculum, although a substantial amount of concern for this study should be found in other subjects such as geography and science.

Measurement is a process whereby numbers are assigned to certain quantitative facets of our environment—namely, those facets which are concerned with magnitude rather than multiplicity. For example, the following questions require measurement for an answer: How wide is this table? How big is a page of this book? How heavy is this rock? The following questions are concerned with multiplicity rather than magnitude: How many words are on this page? How many pupils are in this classroom if there are five rows with six in each row?

The introductory stage of the study of measurement consists of becoming familiar first with the things to be measured (e.g., line segments, solids, weight, and time), and then with such common units of measurement as the inch, foot, yard, mile, pint, quart, gallon, ounce, pound, minute, hour, day, week, month, and year. Familiarity with these things and units of measure is gained through a variety of experiences in seeing and feeling. Initially, comparisons are at a "greater than" or "less than" level. The refinement and complexity of the various units of measure determine the grade in which each one is introduced in the school program.

The analytical stage, which follows the recognition and identification stage, can begin with the development of some fundamental concepts. Length is an essential property associated with a line segment. Measure associates a number with the length of a line segment. The measure of a segment tells how many times a "unit" segment can be fitted into the segment being measured.

In the initial stages, the "unit" chosen is some hand object (e.g., a pencil, an eraser, a piece of string). However, pupils should soon be led to see that while the choice of a "unit" is arbitrary, it is necessary for purposes of accurate communication to choose one or more standard units.

An understanding of the approximate nature of measurement is essential. When a segment is measured, a scale based on the unit appropriate to the purpose of measurement is selected. Every measure is made to the nearest unit. Thus two line segments can have the same measure to the nearest unit even though they may not be exactly of the same length. By using smaller units of measure, more precise measure is obtained.

In addition to the common units of measure the pupil encounters in his immediate environment, he should become familiar with the metric system because of its easy conversion from one unit to another by multiplying by a power of 10 and because it is the system of measure used by scientists.

For convenience in computation, and to express the degree of precision claimed for a measurement, scientific notation is helpful. It is to be expected then that scientific notation, including negative exponents, will be studied before the end of the eighth grade.

The same conceptual sequence used in the measurement of line segments can be followed in the measurement of angles, areas, and volumes. For angles, this starts with developing an intuitive device for comparing the size of a pair of angles. Models of the angles may be made of wire so that they can be superimposed, or one angle may be copied onto another. The basic test for comparing two angles emerging from this approach is: "When two angles have a vertex and side in common and the other side of one angle lies in the interior of the other, the angle whose interior contains the side of the other angle is the larger." Angles may be measured with arbitrary unit angles by "filling in" their interiors with these unit angles. Again the measure is to the nearest unit. Markings on a clock suggest that circles can be divided into equal parts and used to measure angles. This leads to the development of standard unit angles and to the construction and use of a protractor.

For the measurement of area and volume, a start is made by identifying the interior regions of plane figures and solids and developing intuitive means of comparing the sizes of these interiors. In the case of area of plane figures, models can be made of the interiors to be compared and the pupil can see if one can be fitted inside the other, cutting one of the models into pieces if necessary. Similar procedures may be used in the case of solids. Solids may be thought of as containers, and the volume of the solids may be compared by finding which model holds more water or sand. Areas or volumes may be measured with arbitrary units by finding how many of such units are needed to completely fill the interiors of the plane figures or volumes. Graph paper may be used to advantage in finding the areas of plane figures. Pupils should gain an understanding of why standard units in the form of squares or cubes are chosen for measuring area or volume, respectively. The development and use of formulas for the calculation of areas and volumes is particularly important at this time. Measurement of other phenomena such as weight, time, and heat can be studied in a similar fashion. Computation with measures and conversion of units should evolve from actual situations so that it will be done meaningfully and the results interpreted reasonably in terms of significant digits. Ample opportunity should be provided for estimation so that the teacher can determine if the process has meaning for the pupil.

The development in kindergarten and grades one through eight of the measurement strand should lead to such general concepts of measurement as those that follow:

- Measurement is a comparison of the object being measured with a "unit" and yields a number to be attached to the object as the measure of the object.

  (Measurement may thus be conceived as resulting in a pairing of objects with numbers. This pairing is of a type that leads to the notion of a function. Therefore measurement may be treated as a special case of a function. (See discussion of a function.))
- The choice of a measurement "unit" is arbitrary, but standard units are agreed upon for accurate communication and simplified computation.
- Measurement is approximate, and the precision of the measurement depends upon the measurement unit employed.
- Any process of measurement has the following basic properties:

If object A is part of object B, then the measure of A is less than or equal to the measure of B.
[If $A \subset B$, then $m(A) \leq m(B)$]
If objects A and B are congruent, their measures are equal.
[If $A \cong B$, then $m(A) = m(B)$]
If objects A and B do not overlap, then the measure of the object consisting of the union of A and B is the sum of the measures of A and B.
[If $A \cap B = \phi$, then $m(A \cup B) = m(A) + m(B)$]

STRAND 4

Application of Mathematics

Applications in the learning of mathematics are essential in two important ways. First, it is through applications that the pupil is introduced to many mathematical concepts, and his understanding of these concepts is deepened as he applies them to a variety of situations. Second, interest and motivation increase as the pupil sees widespread significant uses of mathematics.

In the first growth stage of mathematical application, a mathematical concept is developed through varied experiences with specific objects and situations that are interesting to pupils. In the early grades, these will be found mostly in the immediate social environment of the pupils. Later the sciences, biological as well as physical, will provide experiences that lead to important mathematical concepts and will also provide new interpretations of previously developed concepts. Finally, it must not be overlooked that as pupils progress through the grades they acquire a fund of mathematical information which itself furnishes the raw material for still other important mathematical concepts.

As the pupil uses a concept in a variety of different concrete situations, its abstract nature gradually evolves. Growth in ability to make generalizations is very important and should not be neglected. Generalizations should not be formalized prematurely. The relationships between a concept and others previously developed and the relationships between concepts and computational algorithms should be observed.

A warning is needed here against confusing mathematical concepts with their interpretations. It is often said, for example, that there are two kinds of division: finding the size of a certain number of equal groups or finding the number of equal groups of a given size from a total number of objects. Actually there is but one division operation, and these are but two of many interpretations which may be given to it. The fact that mathematical ideas are abstract and have no fixed interpretation is the reason for their utility in application.

In a later stage of growth in mathematical applications, the pupil is introduced to concrete problems whose solutions require interpretations of concepts. The pupil starts by observing and studying the problem situation carefully so that the key elements and relationships can be identified and translated into mathematical terms. For some situations this is an elementary task while for others more thought is required. In no instance should the pupil be discouraged from rephrasing the problem situation in his own words.

Having translated the problem into mathematical terms, the pupil proceeds by using previously learned mathematical processes to find a numerical result or a desired formula, which should then be checked for reasonableness.

We recommend against a series of highly formalized problem-solving steps which the pupil can follow without reasoning or understanding. On the other hand, for most of the K-8 curriculum, problems should be sufficiently structured so that the interpretations are clear and some guidance is provided for the solution. This is the type of problem most commonly encountered in texts. A sufficient variety should be provided so that pupils can develop informally the beginnings of some of the more important problem-solving strategies.

Problems that arise in both childhood and adult experience are often unstructured. In instances of this type, the pupil is called upon to do all the preliminary structuring: identifying the relevant aspects of the problem, making the necessary simplifying assumptions, gathering and organizing data, and determining relationships using graphs or tables. Mathematical concepts are utilized through appropriate interpretations and are employed according to principles to find a solution. This is the most desirable form of mathematical application and should be encountered more often by pupils. We recommend that some problems of this type parallel the structured problems throughout the K-8 curriculum, with particular emphasis in grades six, seven, and eight.

The organizing of data involves some knowledge of statistical concepts and procedures. These ideas and processes are to aid in organizing the data so that patterns and relationships can emerge. Therefore, some study of statistical concepts should be included in the program.

In conclusion, a pitfall in using applications in a mathematics program needs to be mentioned briefly. This is the danger of becoming so involved in the details and the study of the application material that the development of mathematical understanding becomes subordinate. For example, in grades seven and eight there is the possibility of becoming so involved in the details of taxation or installment buying, which involve computations of a routine nature, that there is little advancement in mathematical understanding. Only a limited amount of time is available for the study of mathematics in school. A proper balance is needed between the part of this time devoted to applications and the part devoted to mathematics itself.

STRAND 5

FUNCTIONS AND GRAPHS

The function concept is a powerful idea that permeates most of mathematics, and it has many applications. At the elementary school level textbooks and teachers should utilize every opportunity to provide a background of readiness experiences upon which the pupil can develop a correct understanding of the function concept. The function concept can be identified and probably named, since the mathematical usage of the term is very close to its usage in ordinary conversation. For example, we say that the time it takes to drive from San Francisco to Los Angeles is a function of the speed at which we travel, or that crop yield is a function of rainfall.

The idea of a mathematical function has its beginning in the very basic experience of pairing, common in most programs for the primary grades. The small child encounters early the experience of pairing spoken words with objects. He soon learns that more than one object may be associated with the same word. The saying of a word while pointing to an object is sufficient to establish pairing.

In the early beginnings of arithmetic, the pupil also learns to associate a number with a set of objects and to count by pointing to the objects in sequence and pairing them with the set of ordered number words. He learns that counting is a way of determining what number is to be associated with a certain collection of objects; thus, counting numbers may be considered a function of sets. He also becomes aware that different sets of objects may be paired with the same number but that a finite set of objects is paired with one and only one number. A rich background in different types of pairing situations is important. It is important to note that the child may have a wide range of pairing experiences over a long time without being consciously aware of the fundamental idea of pairing.

Possibilities of pairing situations are found in abundance throughout the elementary grades. There are pairing of pupils with their ages, pupils with their test scores, time with temperature, cities with their populations, mountain peaks with their altitudes, and so on ad infinitum. These pairings are often presented in tables and sometimes as graphs, and they should be explored by pupils in order to gain experience with different kinds of pairings. There is an ordered "one-to-one" pairing in which each single object is paired uniquely with a single object. There is also an ordered "many-to-one" pairing in which several objects are paired uniquely with the same object. These one-to-one and many-to-one pairings lead to the concept of a function.

The pairing of geometric figures with numbers, which occurs in measurement and mensuration, involves the function concept. A line segment can be paired with a number which denotes its length. Linear measurement then is a process of determining what number is to be associated with a given line segment. Since many line segments have the same length, this is a many-to-one pairing. Geometric figures may be paired with numbers which may indicate their length, area, volume, perimeter, circumference, radius, or surface area. All of these pairings are of the type that is basic to the idea of a function, and in particular to the idea of a *set function* since geometric figures are sets of points.

The introduction of simple formulas in the arithmetic program affords an opportunity for a deeper experience with the function concept. An equation such as $y = 4x$ can be used to determine a set of ordered pairs of the form (x, y). Some ordered pairs of whole numbers that are members of this set are $(1, 4)$, $(2, 8)$, $(3, 12)$, and so on. The term *function* can now be applied to this set of ordered pairs. Since the equation $y = 4x$ was used in formulating this set, we speak of the function determined by the relation $y = 4x$.

Graphs and tables are effective ways of presenting functions. A start may be made by plotting functions which are sets of ordered pairs of whole numbers resulting in a dot graph. Line graphs of functions may be developed later, and the applications and interpretations of these graphs may be developed also. The concept of a line as a continuous set of points may not be entirely appreciated by pupils until the system of real numbers has been studied, but early readiness experiences are important.

The function concept may be broadened to include mathematical operations. Pupils have encountered binary operations in learning the number facts although the terminology *binary operation* is probably new to them. The basic multiplication facts, for example, describe the binary operation on the set of whole numbers. The operation is binary because only two whole numbers are involved. Thus a binary operation on numbers, such as addition or multiplication, is a special way of pairing two numbers with one number (forming a number triple). The pairings may be presented in a table, such as the familiar addition or multiplication table. In multiplication, the number pair (3, 4) is paired with the number 12, the number pairs (4, 3), (2, 6), (6, 2), (12, 1), and (1, 12) are also paired with the number 12 in the multiplication operation. This indicates that in a binary operation many different number pairs may be paired with the same single number but that a given number pair is associated with only one number.

It is important that pupils have the previously outlined experiences before they have completed the eighth grade. The intuitive experiences will pave the way toward conceiving a function as a set of ordered pairs in which no two pairs have the same first element. The pupil should realize that functions can be presented or described by statements, formulas or equations, tabulated data, and graphs. The pupil is then on his way toward mastering an important concept which is the key to understanding many mathematical ideas that have far-reaching applications.

STRAND 6

SETS

The concept of a set is fundamental for communicating ideas in mathematics, just as it is in our everyday language. We speak of herds, flocks, committees, armies, teams, groups of children, and so on. While it is conceivable that an adequate mathematical text could be written

without using the word *set*, it could not be written without using the concept.

We recommend that pupils become familiar with the simple concepts and language of sets. In kindergarten and grades one through eight the approach should be from a linguistic point of view utilizing the role set terminology plays in accurate and unambiguous communication.

A word of caution: No mention of sets should be made unless an effective use is made of the terminology and concepts in the subsequent mathematical development. In particular there is no place in the elementary curriculum for the development of a *theory of sets* as an end in itself.

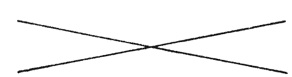
At the first level of introduction, which may be begun as early as in the kindergarten, the teacher uses the terms correctly and encourages their use by the children. There is no need to expect precise mastery by the pupil.

The teacher should talk of *sets* and their *elements* and use the convention of listing the elements of a set between braces, { . . . }. Arithmetic and geometry at all grade levels require extensive use of the basic concept of one-to-one correspondence, commonly called matching. This concept relies implicitly, if not explicitly, on a preconceived notion of set to describe what is being matched The abstract nature of the number four can be learned by considering many different sets and by noting that the elements of some sets can be matched in a one-to-one correspondence with the elements of a particular set, $\{\bigcirc, \triangle, \varnothing, \square\}$, to which we assign the number four.

In presenting arithmetic and geometry, the teacher should talk of set operations and use the correct symbols: *set union* ($\cup$), *set intersection* ($\cap$), and *relative complement* ($-$). Examples of these are found:

— in addition	2	+	3	=	5
(set union)	{x, y}	$\cup$	{a, b, c}	=	{x, y, a, b, c}
— in subtraction	5	−	3	=	2
(relative complement)	{a, b, c, x, y}	−	{a, x, y}	=	{b, c}

— in geometry



(set intersection) "Two different straight lines intersect in, at most, one point."

The concept of a *subset* of a set is equally natural. For example, "The set of even numbers is a subset of the set of all numbers," or "A line segment is a subset of a line."

At the second level of using set language, the pupil will be expected to use correctly the terms his teacher used at the first level. His teacher at this level will use these terms in more complex situations and introduce some new concepts: *ordered pair* [(a, b)], *cartesian product* ($A \times B$), and the *notation for sets,* which incorporates a description of the elements in the set, $\{ \times \mid$ "condition on $\times$" $\}$.

A definition of multiplication which is particularly appropriate for multiplication topics in grades four, five, and six can be given in terms of a cartesian product of two sets—that is, as the number of items in a rectangular array. Thus 2×3 is the number of elements in an array of two rows and three columns as shown on the right. We might label the rows 1 and 2; the columns a, b, and c. Then the *'s in the array can be labelled as ordered pairs:

$$\left\{ \begin{matrix} (1,a) & (1,b) & (1,c) \\ (2,a) & (2,b) & (2,c) \end{matrix} \right\} = \{1, 2\} \times \{a, b, c\} \qquad \begin{matrix} * & * & * \\ * & * & * \end{matrix}$$

The number plane may be introduced in geometry. The location of points on the plane is made by assigning an ordered pair of numbers to each point.

There is also an important use to be made of sets in helping the pupil to distinguish between properties enjoyed by some sets of numbers and not by others. As is pointed out in discussing the strand on numbers, our number system evolves from the set of counting numbers to the set of all real numbers. In introducing "new" numbers, a new *set* of numbers is created, and we must carefully distinguish between the properties one set possesses but another does not. Although a formal study of negative numbers is not recommended prior to grade seven or eight, readiness for negative numbers can be promoted in earlier grades as opportunities occur. For example, suppose a pupil asks, "What happens if I subtract 3 from 2?" The pupil should be led to see that there is no natural (counting) number which answers this question and that hence the set of natural numbers is not closed under subtraction. The teacher might conclude her remarks with the statement, "Later you will construct a new set of numbers, larger than the set of natural numbers, which will be closed under subtraction. In that system you can subtract 3 from 2."

In the final developmental stage of the language of sets in kindergarten and grades one through eight, we expect the pupil to use these

set concepts and terms correctly and with reasonable precision They will play an important role as he studies inequalities, solution sets, informal geometry, and the application of mathematics to the social world, to the technological world, and to mathematics itself.

The article, "Operating With Sets," by E. J. McShane in *Insights Into Modern Mathematics,* Twenty-Third Yearbook of the National Council of Teachers of Mathematics, is a rather detailed treatment of sets.

STRAND 7

THE MATHEMATICAL SENTENCE

Since language is the primary means for the communication of ideas, a pupil should develop skills in using language to express ideas with clarity and precision. The language of mathematics is particularly suited to this purpose. The pupil's experiences with mathematical sentences should grow naturally out of his experiences with arithmetic. A beginning is made when a pupil recognizes and can write or say the numbers which tell how many objects there are in a set. The addition operation is conceived as the joining of two nonintersecting sets with a known number of objects in each and telling how many objects there are in the resulting set. If there are four members in one set and two members in a second nonintersecting set, the mathematical sentence is: four and two is equal to six. Many experiences with pictures and objects showing the joining of two sets will lead to the basic sentence form: ---- plus ---- equals ----, where pupils insert the numerals which tell the story. The symbols for plus, minus, and equals are later used as replacements for words in this sentence form. Sentences involving the other operations are similarly developed through concrete or vicarious experiences.

With this background a pupil learns that a mathematical sentence is simply a statement concerning numbers. A mathematical sentence may employ such nouns as number, set, or point. Replacements for these nouns may be numerals or variables such as geometrically shaped frames or other symbols. The term *variable* need not be used with pupils in introductory stages and should not be used without adequate previous intuitive experience, although the teacher should have an understanding of the concept and use of a variable. A mathematical sentence also employs a verb form or its symbol such as is ($=$), is not ($\neq$), is less than ($<$) or is greater than ($>$). Conjunctions such as $+$, $-$, $\times$, and $\div$ also occur, and the noun-verb-noun sentence form is common.

The use of mathematical sentences in elementary instruction has in the past been handicapped by inadequate notation. If a sequence of various operations is involved, then symbols such as the various forms of parentheses ( ), [ ], and { } are needed to specify the order of operations.

Thinking through a problem situation often results in the formulation of a mathematical sentence. When a pupil learns to describe a problem in his own words and then to state the relationships by use of a mathematical model in sentence form, he has a sound approach to problem solving. The process of building a mathematical model should not be mechanical but should lead to a deep understanding of the mathematical process. Pupils who understand and use this approach to problem solving seldom ask, "Do I multiply or divide?" A problem situation may be described with a single statement or with several statements as a series of equations or inequalities.

Initially, mathematical sentence structure can be presented very simply. If the problem situation is one in which the pupil first adds and then subtracts, he can state each procedure in a separate sentence or the two procedures in a single sentence. Furthermore, a single mathematical sentence can be used to unify different problem situations. For example, the so-called three cases of percent can be unified by the sentence $a \times 1/100 \times b = c$, where a represents the percent, b the base, and c the percentage. We may solve for any unknown element in this sentence if we are given the values of the other two elements.

Mathematical sentences are particularly useful in studying the properties of operations by providing a form for expressing such principles as commutativity, associativity, and distributivity. Sentences can also show the "doing" and "undoing" of inverse operations, as for example: $(8 + 3) - 3 = 8$ and $(24 \div 6) \times 6 = 24$. Sentences can provide a detailed analysis of certain operations involving fractions. For example: $5 + (2/3 + 1/2) = 5 + (4/6 + 3/6) = 5 + 7/6 = 5 + 1 1/6 = 6 1/6$.

Mathematical sentences may assist a pupil to make generalizations in the early stages of his mathematical education. If he observes that $0 + 1 = 1$; $0 + 2 = 2$; $0 + 3 = 3$; and so on, he should be able to generalize that for every whole number n, $0 + n = n$.

Much of a person's mathematical thinking involves equations and inequalities, so a clear understanding of the meaning of equality and inequality needs to be acquired. This and other questions of form of mathematical sentences are intimately related to logic and are discussed more fully in the following section on logic.

It is very important that a textbook sequence provide a developmental arrangement of experiences explicitly designed to help the pupil acquire the ability to translate problem situations and facts in the language of mathematics and to acquire familiarity with mathematical sentences.

STRAND 8

LOGIC

It should be observed at the beginning of this discussion that we do not suggest a formal study of logic in the elementary school. Rather at this level we are concerned with building a strong background of readiness experiences in logical thinking which will facilitate the pupil's acquisition of patterns of precise mathematical reasoning. This is accomplished by utilizing opportunities in the arithmetic program to delineate logical ideas and concepts.

Logic is fundamentally the grammar of mathematics. It provides a way of organizing mathematical ideas and of clarifying their meaning. An important objective of mathematical instruction is showing how knowledge may be organized and made increasingly effective. Logic provides the grammar for this structure and enhances the effectiveness of the mathematical ideas. Without logical thinking, learning would result in a hodgepodge of information and mechanical techniques.

In the following discussion we present some concepts of logic with a few comments on how they may be informally encountered by pupils.

At the elementary school level, the matter of giving reasons is largely a one- or two-step sequence. Usually there is an agreed upon principle or definition which can be given as an immediate justification for a particular case. $37 \times 0 = 0$ is true because it is an instance of an accepted general principle that any number times zero is zero. In multiplying 23 by 3, it is possible to multiply the two tens and three ones separately by three through application of the distributive principle. $3 \times (20 + 3) = (3 \times 20) + (3 \times 3)$. The concept of logic involved in these examples is that all instances of accepted or proven generalizations are true. This concept enables the pupil to relate all the particulars of arithmetic computation to a few key principles, and it is a concept which can be left at the intuitive level for elementary school pupils.

The concept of identity in logic clarifies the meaning of the equals sign (=), for the equals sign is the mathematical symbol for "is identical with." For example, $3 + 2$ designates a number, and $4 + 1$

designates the same number. This fact is expressed by $3 + 2 = 4 + 1$. Similarly, $3/7 = 6/14$ because $3/7$ and $6/14$ designate the same number. The principle that we may use any one of several names or designations for an object is used throughout the arithmetic program. Thus $1/2 + 3/7 = 7/14 + 6/14$ because $1/2$ and $7/14$ designate the same number, and $3/7$ and $6/14$ designate the same number. Textbooks and teachers should stress correct use of the equals sign throughout the elementary school program.

Definitions are the statements which establish the meanings of new terms by using primitive (undefined) or previously defined terms. Certain growth stages can be identified in connection with defining new terms. At first, a new term can be introduced through examples. For instance, the term *less than* can be introduced by showing that two is less than five through matching one-to-one the objects in a set of two objects with the objects in a set of five objects resulting in the latter set having unmatched objects. Next, the *less than* symbol ($<$) can be introduced and described in operational terms. For example $2 < 5$ means that two is named in the counting sequence before five or that two is to the left of five on the standard drawing of the number line. A start towards a mathematical definition is made when it is developed that $2 < 5$ means that there is a counting number that can be added to two to get five. The formal definition will have to wait until more precise algebraic terminology is available. Throughout the elementary grades, however, pupils should have abundant opportunities to develop and apply informal definitions.

In the early grades, pupils become acquainted with the use of the words *all* and *some*, which are known as quantifiers in logic. The pupils are able to distinguish between the meaning of the two statements, "all balls are red" and "some balls are red." In mathematics the word *some* means "there is at least one." Statements about the properties of numbers require quantification because certain properties are possessed by all numbers while other properties are possessed only by some numbers. Quantification should be informal at first as in the statements: "any number times zero is zero" and "what number added to itself equals itself." Later the quantification may become more formal: "for each number n, $n \times 0 = 0$" and "there is a number n such that $n + n = n$." To show that a statement like "all odd numbers are primes" is false, it is necessary only to find one odd number that is not prime. This number is called a *counter-example* in logic. Pupils take great delight in this sort of activity.

Terms such as *and*, *or*, *if—then*, and *not* are important logical terms and should be informally introduced in the elementary grades. Pupils should be able to distinguish between "numbers that are odd *and* prime" and "numbers that are odd *or* prime." The meaning of the *if—then* sentence can be analyzed in simple situations such as this: "If Helen goes to the movies, then Jim goes," and noting that this sentence would be false if Helen went to the movies and Jim did not go. The *if—then* logic sequence can be further extended in the upper grades to geometric applications such as in testing the accuracy of the statement, "If a plane figure has four equal sides, then it is a square."

Logic is also concerned with developing other types of patterns for drawing valid inferences. Simple inferences schemes should be informally introduced and kept at a level that makes sense to pupils. For example, a situation like the following might be posed. Suppose it is agreed that any whole number is either odd or even, and suppose that a particular whole number is not even. What conclusion can be drawn?

Still other opportunities for developing patterns of logical reasoning lie in the many arithmetic situations where deductive reasoning may be applied. For example, "If $2/3$ of a number is 6, then $1/3$ of a number will be $1/2$ of 6 or 3. Hence, the number will be 3×3 or 9." It is important to keep in mind at any level that establishing patterns of logical thinking provides the key to the learning of mathematics.

Part Two

MATHEMATICS PROGRAMS FOR TEACHERS

The success of the mathematics program that is based on the essential strands of mathematics depends in large part upon the mathematical competencies of elementary school teachers. Research continually points out that the preparation of teachers in mathematics is the important prerequisite of an improved mathematics program; also that poorly prepared teachers induce in too many young people an enduring fear and distaste toward mathematics. It was with these thoughts in mind that the Subcommittee on Implementation of Teacher and Pilot Programs started several studies centered chiefly in the preservice and inservice mathematics education of elementary school teachers. It was the contention of this subcommittee that mathematics education should (1) inspire teachers with confidence in their ability to teach mathematics; and (2) endow teachers with respect and appreciation for mathematics.

Elementary school teachers are not being blamed for a lack of mathematical knowledge. Neither are teacher-training institutions being blamed for not having given elementary school teachers a better background in mathematics. Since elementary school teachers must teach many subjects, their educational preparation places a heavy demand both upon the teachers and upon teacher-training institutions. The present shift in mathematical emphases set forth in Part One makes it imperative that teachers have a much more extensive knowledge of mathematics than heretofore. This means that both preservice and inservice training programs for teachers must be strengthened.

Preservice Training Programs for Teachers

To initiate its study the subcommittee investigated mathematics requirements for elementary school teachers in ten representative California colleges and universities. It found that in nine of the ten institutions it was possible to complete graduation requirements in elementary education with no course work in mathematics except as it was taught in education courses. At the same time a teaching credential could be obtained directly from the California State Department of Education on the basis of a recognized degree with no college courses in mathematics. This condition has been corrected by the new credential requirements that are now effective.

The subcommittee's study of the elementary teacher-training programs in mathematics at the ten institutions revealed that no mathematics methods courses were offered by two institutions, a composite methods course was offered by one, a mathematics methods course was offered by two, a content course in mathematics was offered by three, and both a content course in mathematics and a methods course were offered by two. This information makes it apparent that the teacher-training programs available are lacking in uniformity. The subcommittee also found that the members of the mathematics and education departments of the schools did not generally agree that more course work in mathematics should be required for the elementary teaching credential, nor on other changes that might be made.

Changes in preservice mathematics education were viewed by the subcommittee to be dependent upon the following considerations:

- A clear definition of the mathematics curriculum for the elementary and secondary schools
- An arrangement of courses for teachers evolved from closely woven relationship among essential content, applications, relevant aspects of the psychology of learning, and of the growth and development of youth
- A clear understanding of the knowledge and skills a teacher needs to teach mathematics
- A feasible training program of education in mathematics that teacher training institutions can implement smoothly and without undue delay
- A carefully developed research program directed toward the improvement of teacher education as a whole

The Advisory Committee on Mathematics endorsed the recommendations for the training in mathematics of elementary and secondary school teachers proposed by the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America.[1] These recommendations, prepared by mathematicians, have received critical consideration throughout the nation. Level I recommendations are for teachers of elementary school mathematics. These specify at least two years of college mathematics study consisting of the following courses:

- A two-course sequence devoted to the structure of the real number system and its subdivisions

[1] "Recommendations of the Mathematical Association of America for the Training of Teachers in Mathematics." *The Mathematics Teacher*, LIII (December, 1960).

- A course devoted to the basic concepts of algebra
- A course in informal geometry

Level II recommendations pertain to teachers of the elements of algebra and geometry. Level III recommendations pertain to teachers of high school mathematics. In all instances the recommendations call for teachers of mathematics to include in their college preparation courses that provide them with substantial increases in mathematical knowledge over that previously necessary. The Advisory Committee on Mathematics further recommended that the specifications for levels I, II, and III be incorporated in credential requirements.

The Advisory Committee on Mathematics approved the recommendation of the Subcommittee on Implementation that elementary teaching candidates have adequate preparation in the methodology and psychology of teaching mathematics in the elementary school.

A further recommendation of the Subcommittee approved by the Advisory Committee was that each college and university assess the competencies of each member of its department responsible for teaching mathematics to prospective teachers from the following points of view:

- Adequacy in understanding both traditional and modern content
- Adequacy in inspiring young people and inservice teachers
- Adequacy in experience within the elementary or high school setting

Where inadequacy is noted, every effort should be made to rectify the matter. Each prospective teacher deserves the finest possible mathematical preparation from those who are themselves master teachers and who are conversant with the problems to be faced by the classroom teacher.

A final recommendation bearing the approval of the Advisory Committee was that the State Department of Education sponsor a series of conferences involving representation of education, mathematics, and administrative departments of each teacher-training institution in California. Major purposes of the conferences would be (1) the dissemination of information concerning needs in mathematics for elementary and secondary credential candidates; and (2) the development of a working agreement among the representatives of each staff as to an immediate proposal to its own staff.

Inservice Education Programs

A summary of recommendations concerning teachers in effecting changes for mathematics instruction follows.

SPECIFIC GOALS

The focal point of inservice programs for elementary school teachers should be towards understanding the strands of mathematical concepts. Although the period through 1966 is one of critical importance due to the introduction of new textbooks, there is also need for a long-term program. In addition to the study of mathematical concepts, attention should be given to classroom techniques to foster pupil discovery of mathematical ideas.

GENERAL COORDINATION

Inservice education activities can benefit from statewide coordination and cooperation. Personnel now available can be used to advantage for this purpose, and in doing so, needless duplication of effort can be avoided. The California State Department of Education is recommended as the coordinating agency, provided additional staff is made available to do the required work.

PLANNED PROGRAMS

School districts should have long-range plans for on-the-job teacher education in mathematics. Approval of requests for consultant service from and through the Department of Education should be contingent upon such plans. Funds to finance related research projects should be sought.

IDENTIFIABLE RESPONSIBILITY

Offices of county superintendents of schools as well as school districts will require personnel having an assignment to coordinate mathematics inservice education. An allotment of man-days should make the assignment feasible. An urgent first task would be to develop or place a teacher-specialist in each elementary school who may assist with the inservice education program.

WORK WITH PARENTS

Parents expect and need to understand the instruction given to their children in elementary schools. The time schedule for informing parents should parallel the one for the program for teachers. This may be accomplished in part through adult education classes.

THE TEACHERS' TEACHER

The teacher for teachers must not only be qualified in mathematics but must also understand the elementary school program and its function, know the nature of the elementary school child, and establish his position as a colleague of teachers who shares their goals and values. Individuals with these qualifications should be encouraged to become teachers of teachers.

ADEQUATE TIME NEEDED

Inservice education requires a reasonable allocation of teacher time. Present observations support use of at least 30 hours for instruction on number systems and geometry, the areas of instruction to be selected for relevance. Additional time will be required to meet the special emphases of each grade level. Discussion is an essential portion of teacher instruction; classroom observations can illuminate pupil discovery procedures. It is judged valuable for well-prepared and experienced teachers to engage in a phased program of receiving inservice education and offering related instruction to their classes. Inservice education programs that are too brief may fall far short of the goal needed.

SUPPORTIVE ACTIVITIES

Mass media techniques—e.g., television and programmed textbooks—may prove useful adjuncts of the basic face-to-face instruction, but should not replace it.

EQUITABLE PERSONNEL POLICIES NEEDED

Since an inservice education program of adequate dimensions requires the participants to spend approximately the amount of time required to complete a two-unit graduate course, participation in the program is a requirement that goes beyond the professional activity that may be rightfully expected of teachers. If they are required to participate in an inservice program, they should be freed from other responsibilities for sufficiently long periods of time to do so or be rewarded adequately for the time they spend in the program.

THE SETTING FOR EVALUATION

There must be a realistic expectation regarding the level of competency in teaching mathematics that teachers may be expected to attain. The heavy and varied demands upon elementary teachers entailed by the regular curriculum have in the past made it impossible for teacher-training institutions to give special attention to mathematics and science instruction in the teacher-training program and in all general education preceding it. Teachers can be expected to reject any program in which the administration and other responsible persons

fail to exhibit a sympathetic understanding of all facets of their situation and need.

Studies of Inservice Education Programs

The emphasis upon leading students to discover and understand the underlying concepts of mathematics in addition to the fundamental mathematical techniques requires competent teachers at all levels who understand and appreciate both the traditional and modern points of view. Teachers of mathematics need the opportunity through carefully planned inservice programs to strengthen their mathematics background.

A basic question is: How can a school system organize an inservice program suited to the needs of its teachers? Allied with this are these questions: Where does one begin? Whom does one involve? How does one proceed?

There are as many answers to these questions as there are successful inservice programs. In attempting to set up a program for a particular school system, one can look to other similar school systems that have programs in operation to find those common elements suggestive of approaches having high probability of success.

An overview of several inservice programs in operation points up the following common elements:

- *Administrative support.* The administration must appreciate the need for the program and support it.
- *An initiating individual.* Someone must take an active leadership role to build enthusiasm, organize materials and meetings, and generally see the whole program through to completion. Behind many successful programs may be found the influence of one key individual.
- *Definite materials.* Teachers seem to respond better when they know that the materials they are reviewing are going to be used by them in the classroom.
- *Definite schedule of "classes."* The teachers meet regularly and know in advance what they can anticipate for each session.
- *A consultant.* Most teachers need someone whose knowledge of mathematics they can respect to guide them through the rough spots of the materials.

Four Pilot Inservice Education Programs

In order to ascertain further the specific and essential elements of successful inservice mathematics education programs, there was under-taken at the request of the Advisory Committee on Mathematics a study of four inservice programs in operation during 1961-62 and partly supported by funds under the National Defense Education Act. The four programs studied were being administered by (1) Mt. Diablo Unified School District; (2) Whittier Union High School District; (3) the office of the Stanislaus County Superintendent of Schools; and (4) Orange County school administrators.

The programs were studied to secure answers to the following questions:

- What environmental factors influence the teaching of new ideas to adults?
- What organizational matters must be given attention in an inservice program?
- What incentives should be established for teachers?
- What procedures for leadership procurement should be followed?
- What are further inservice needs following a year's program?
- What generalizations can be reached based on the studies?

Descriptions of the Programs

After the following overview of the four programs have been studied, some significant generalizations with regard to the above questions are presented.

MT. DIABLO UNIFIED SCHOOL DISTRICT

The Mt. Diablo Unified School District, comprising 31 elementary schools, six intermediate schools, and six high schools, has as one of its goals the development of a sequential, well-articulated program in mathematics for kindergarten and grades one through twelve. The District Mathematics Steering Committee has representatives from all grade levels.

A program of acceleration and enrichment inaugurated in 1958 served as a catalyst for stimulating teacher interest in the desirability of acquiring additional mathematical background through college courses and district inservice education programs.

A group of 16 elementary teachers with some mathematical background and a willingness to meet for two hours biweekly with a mathematics consultant were chosen to teach School Mathematics Study Group (S.M.S.G.) materials the first year (1961-62); 32 teachers were added for the second year of the program. Plans call for increasing the number each year so that the teachers in each elementary school will be competent to teach the new mathematics by the time newly adopted textbooks are put into use.

At each meeting the mathematics consultant presents the material to be covered through a seminar-type approach. A sharing of ideas is encouraged. The project director assists by showing how the material may be presented effectively to students. District administrators are encouraged to observe in classrooms where the material is being used. Teachers are selected on a district geographical basis and on a basis of mathematical competency. The goal is to have key teachers in each school of the district.

SCHOOLS IN STANISLAUS COUNTY

Stanislaus County has about 40,000 pupils in kindergarten and grades one through twelve and 1,400 full-time junior college students. In this county 20,000 students are in two city school districts; a like number are in 31 elementary districts, six high school districts, and one unified district. The two city districts under a joint administration cooperatively have developed and evaluated curricula in mathematics.

In 1955 the Stanislaus County Board of Education approved a study of elementary arithmetic curricula. This study resulted in teachers guides for arithmetic for kindergarten and grades one through eight. In 1961 the Board again approved a four-year program of improvement to meet the changing needs in mathematics education. Needs identified by the administrative and supervisorial staff in the office of the County Superintendent of Schools included (1) experimentation with various newer programs; (2) articulation of programs at the intermediate and high school grades; (3) organization of special programs for the gifted; (4) development of inservice education programs for all teaching personnel within the county; and (5) orientation of the public to the philosophy, needs, and practices in mathematics.

- Eight teachers in six districts used School Mathematics Study Group fourth grade materials as basis for inservice education. Ten meetings, each for one hour, were held with an outside consultant in charge.
- Ten teachers in one large elementary district used S.M.S.G. seventh grade materials as a basis for inservice education. The one-hour meetings were held with an outside consultant in charge.
- Fourteen seventh and eighth grade teachers in seven elementary districts used S.M.S.G. seventh grade materials as a basis for inservice education. Thirteen meetings, each for one hour, were held with a high school teacher in charge.

- Sixty-two teachers representing a countywide coverage of school districts attended from one to four meetings to review the status of the arithmetic program in Stanislaus County and to receive limited instruction in modern approaches to mathematics.
- Seven exploratory-type conferences were held at the secondary level.

WHITTIER UNION HIGH SCHOOL DISTRICT

The Whittier Union High School District operates on an 11-month employment plan for teachers which provides 14 working days prior to the fall opening of school for inservice activities. In 1961 these included eight, three-hour section meetings of four different groups that were devoted to the study of modern mathematics materials. The four groups were pre-algebra, terminal mathematics, algebra, and geometry. Each of the groups then met 12 times at regular intervals during the school year; 52 teachers were involved. A mathematics professor from a local university served as consultant.

A three-hour county institute on mathematics was devoted to the study of topics pertaining to curriculum articulation, modern instructional materials, and inservice education. Inservice education plans for the future were developed.

Also a series of six meetings on modern mathematics was held for district administrators during the fall of 1961. Twenty-eight administrators, including three from elementary districts, attended. The nature of changes in modern school mathematics was discussed, and the participants were given opportunity to become acquainted with some of the vocabulary of the new mathematics and with the methods used in presenting new ideas and concepts.

ORANGE COUNTY SCHOOLS

In 1961 the office of the Orange County Superintendent of Schools planned an inservice education program specifically for administrators and supervisory personnel. The project was allied with the work of the California Advisory Committee on Mathematics as an action research project in mathematics inservice education. The project's specific objective was the education of principals, supervisors, and curriculum workers in the new mathematics. It was planned that such leaders would then in turn be prepared to devise inservice programs for teachers at the local level. Thirty days of consultant service were made available to the schools in the county through the use of NDEA funds.

The two principal considerations that guided the study appear on the next page.

- What inservice education plan will be most effective in offering to principals and supervisory personnel sufficient background in newer mathematics to prepare them for leadership roles necessary for directing inservice education programs in mathematics for elementary teachers?
- How may principals and supervisory personnel best meet the increasing demands for leadership in mathematics curriculum development and the improvement of classroom instruction techniques in mathematics?

The inservice program featured the following activities:

- Four seminar sessions were devoted to the basic principles and techniques characterizing the new mathematics. Each session included discussion of ways the ideas presented might be used by the participants with local teacher groups. Plans for simple experimental studies at the classroom level were suggested and the leadership roles of supervisory personnel at the local level were stressed. Each of the four seminar sessions was scheduled in a participating district with this district serving as host.
  - A series of four classroom demonstration conferences was conducted, each one related to the seminar sessions. Each conference featured four to seven demonstration lessons taught with classes from grades one through six.
  - Seminar participants received a series of critical reports from authorities on national curriculum studies in mathematics.
  - A panel of consultants from nearby colleges and universities was made available for work with teacher groups in the districts sponsoring demonstration-conferences.
  - Twenty-two of the 31 elementary districts in the county responded to an invitation to participate in the project by sending from one to four representatives to the several meetings. There were 80 participants.

The following seven items identify project outcomes for one or more of the participating principals and supervisors:

- Motivation for further independent study in mathematics and methodology in teaching
- Increased understanding of the new mathematics
- Increased competence in designing and preparing experimental

studies in curriculum and methods of classroom instruction in the field of mathematics

- Increased competence in the effective classroom utilization of mathematics materials
- Increased knowledge of how to meet individual differences among pupils, including ways to challenge the more able learner and the gifted child
- Increased capability to conduct inservice education programs for elementary teachers in "new" mathematics and methods of teaching
- More effective curriculum development in mathematics at the district level

Other Inservice Programs in Mathematics Education

In addition to the detailed study of the four inservice programs described in a previous section, a more cursory review was made of several other inservice programs. Several significant observations are to be noted from this survey:

- There is considerable variety in types of inservice programs and the organizational procedures under which they operate.
- While programs vary in organizational features, a commonness of purpose is evident. The upgrading of mathematics curricular programs with particular emphasis on the strengthening of teacher background and competence in the new mathematics are objectives of all programs.
- The financial assistance available under the provisions of the National Defense Education Act is a vital factor in the successful operation of the majority of inservice programs.
- Evidenced in many programs is the decided effort to make increasing use of locally identified leadership in conducting inservice programs and generally giving help to teachers interested in their own reeducation.
- The best results grow out of those programs that are carefully planned and feature not only the support but active involvement of administrative and supervisory personnel in addition to the teacher group.
- The inservice education experience is enhanced when teachers have opportunity to see new materials actually being taught in demonstration sessions followed by free discussion on observations made.

The following programs sponsored by bureaus of the California State Department of Education were studied:

- Three regional mathematics workshops for teachers and administrators, particularly those in the business area of the curriculum—Bureau of Business Education
- Three regional conferences on mathematics, a one-week inservice workshop in elementary school mathematics, and three summer conferences emphasizing local leadership training in mathematics inservice programs—Bureau of Elementary Education
- Six two-day regional conferences on the teaching of mathematics—Bureau of Secondary Education

In addition the Committee also studied the inservice education programs that were being conducted by schools under the office of Santa Barbara County Superintendent of Schools and those being conducted in the following school districts:

Fresno Unified School District
Grossmont Union High School District: Heartland Articulation Committee (teachers from elementary districts within the high school district)
Hilmar Unified School District
Millbrae Elementary School District
Modesto City Elementary School District
Morongo Unified School District
National Elementary School District, San Diego County
Palo Alto City Unified School District
San Francisco City Unified School District
San Diego City Unified School District
San Juan Unified School District
Sweetwater Union High School District, San Diego County
Whittier City Elementary and Union High School districts

To meet the tremendous need for more inservice mathematics education throughout the state, it is hoped that there will be a rapid expansion in the number of districts offering programs. Funds presently available under the National Defense Act Administration are not adequate to meet the requests for financial assistance. In fact, they can do little more than scratch the surface in meeting the needs that exist. The inservice education problem will be solved on a statewide basis only when all educational agencies cooperate in meeting the need.

Recommendations for Inservice Education

The recommendations made by the Advisory Committee on Mathematics as to concepts, content, and pedagogy to be contained in the next elementary arithmetic textbooks will need implementation procedures of great magnitude. Though nearly all of the topics now taught will be unchanged, the recommended emphasis upon the structural aspects of the fundamental concepts underlying the arithmetic curriculum will reflect profound change. The reeducation of most elementary teachers now in service will be an immediate necessity if such a program is to be in action by 1965. Also, the impact of the recommendations will be felt strongly in programs of secondary school mathematics, and the reeducation of teachers there must keep pace with changes triggered by the emerging elementary curriculum.

The Subcommittee on Implementation has agreed that a sensible approach to this problem is one which utilizes all existing professional agencies available to teachers in California schools. The Committee also agreed that the coordinated use of resources is imperative in order that valuable services and time will not be wasted in the period before new textbooks are available and new mathematics programs are in operation.

The Subcommittee stated that the objective for any reeducation program must be to bring the education of teachers now in service up to the quality recommended by the Subcommittee in its report on preservice education presented to and approved by the Advisory Committee. Furthermore, the Committee agreed that reeducation should be provided only by those qualified both in mathematics and in the psychology of the learning process.

To these ends the Subcommittee proposes a three year inservice program coordinated at the state level and operated at the district level so that the existing needs of each district may be met. Attainment of the goal now planned would result in a sequential mathematics program for kindergarten and grades one through twelve being in full operation in 1965, one that would provide for the development of the mathematically literate and proficient citizens needed in our culture.

The Subcommittee urges the State Department of Education to seek staff and funds to carry out its part in this program.

A Proposed Statewide Plan

(A Responsibility of the State Department of Education)

PHASE I

The services of the Advisory Committee on Mathematics should be continued and extend through a permanent though small group that includes industrial representation.

Functions of the permanent advisory committee for mathematics include the following:

- To continue study toward recommendation of strands for secondary schools
- To give overall leadership for inservice education
- To receive and evaluate major experimental plans
- To evaluate progress in each aspect of instructional improvement
- To maintain liaison with the California State Curriculum Commission on instructional materials

A team of no fewer than three full-time resource persons from the field of education should be appointed to the permanent advisory committee to represent the elementary and secondary schools and colleges. One should be a mathematician and one a general elementary educator with a background in mathematics. The period of service for team members would be at least two years, preferably three years. The members should incur no loss of salary; their assignment should be in the Division of Instruction, California State Department of Education. Existing personnel provided through NDEA funds should be maintained for general service.

Functions of full-time resource personnel would include the following:

- To serve as a resource for, and liaison with, agencies such as the State Curriculum Commission and the bureaus in the Division of Instruction
- To coordinate inservice programs for teachers of kindergarten and grades one through eight throughout the state in order to secure reasonable uniformity in the endeavor to conduct the activities required to further the recommendations of the Mathematics Advisory Committee.
- To prepare recommendations for supplementary inservice programs to be carried on through the use of media such as television and programmed learning.
- To provide the major teaching services for regional summer institutes

Preparation should be made for regional summer institutes in conjunction with the Advisory Committee on Mathematics. These institutes should be operated in sessions for (1) administrative and supervisory personnel for two to four weeks; (2) selected teachers for four weeks; and (3) mathematics specialists for six weeks. The institutes would serve the following purposes for each group.

- The administrators and supervisors would participate in the following activities:
  - Studying the recommended mathematical content for kindergarten and grades one through eight or grades nine through twelve, with emphasis on grade placement of concepts, the depth of development of concepts at each grade level, and the teaching objectives of the new material (manipulative skills with understanding, precise language, and the like)
  - Examining the teaching techniques which are an integral part of the change in the program of school mathematics

Studying some recommended structures and procedures for implementing the change to modern materials in mathematics

Studying experimental programs which may contribute to future developments in school mathematics

- The teachers would participate in the following activities:

Studying in detail the Strands of Mathematical Concepts in order to understand them and to be proficient in their use

Developing adeptness in teaching mathematical concepts presented in the Strands

Developing classroom procedures for experimentation and demonstration to be used during the ensuing year

- The mathematics specialists would participate in these activities:

Studying content selected for presentation to teachers grouped from each of the grade levels (kindergarten and grades one through three, grades four through eight, and grades nine through twelve)

Giving attention to the content presented in the preceding grade and in the next higher grade in relation to that taught in each grade

Studying structures for inservice education programs, including goals, problems, and probable accomplishments for different programs

Determining resources to be used by the specialist in the inservice program

Evaluating procedures for stimulating action by teachers, administrators, and school boards

Planning effective methods of reeducating teachers

Examining the nature of the learning process as it relates to students at all levels

Teams of consultants composed of personnel from elementary and secondary schools, colleges and industry should be established on a regional basis to meet on call with boards of education, citizens committees, administrative groups, P.T.A.'s, and the like. These teams would provide expert advice to aid and support the specialists in local inservice programs by reinforcing the importance of a modern school mathematics program from several points of view.

The establishment of regional depositories of materials (films, modern texts, evaluations of experimental programs, recommendations of various groups, tapes, and the like) to support inservice education should be studied. Such materials could be drawn upon by mathematics specialists and consultants.

In addition, the State Department of Education should provide the following services:

Develop a plan for evaluation of inservice education at each state giving special attention to changes in behavior of teachers.

Work with offices of county superintendents of schools and large school districts to relate their needs to plans at the state level. The nature of regional studies should be publicized at an early date to promote coordination. Inservice requests under NDEA should be approved only as they lend support to the goals of the total program and do not duplicate services or time.

Formulate or modify accreditation procedures to aid the schools in evaluating their mathematics programs and the mathematical background of teachers of mathematics in light of the recommendations of the Advisory Committee on Mathematics.

Work with institutions of higher learning to evaluate the ability of their newest graduates to understand and to teach the concepts recommended by the Advisory Committee.

Study the development of effective techniques to be used in explaining the goals and progress of the statewide program to the Legislature, school boards, and general public.

Invite and coordinate the participation of industry and commerce in the reeducation effort. Industry and commerce have provided personnel, services, supplies, and financial aid such as scholarships for participants in very similar programs.

Develop proposals for sufficient financing to support the active phases of the inservice effort.

PHASE II

Provide regional summer institutes and follow-up service programs.

Experiment with mass media techniques to implement regional inservice programs during the year.

Develop evaluation techniques to ascertain the effectiveness of the inservice program.

Continue to employ effective techniques in explaining the goals and progress of the statewide inservice program to the Legislature, school boards, and the general public.

PHASE III

Continue inservice endeavors of the first two years to include all regions not yet served.

Utilize evaluation techniques to ascertain the effectiveness of the second year.

Document the three-year study in all its aspects.

Each school district has many responsibilities that may be met by direct action of its board or through coordination of its efforts with that of the county superintendent of schools. Each district administrator must discover the levels of understanding and ability of the teachers in the district to apply the strands of the mathematical concepts. To the degree that reeducation is needed, each district should commit itself either to join a statewide inservice program or to provide an adequate opportunity with its own resources. In either instance, released time for study, in-lieu-of-credit for salary for salary purposes, adequate materials, reasonable assignments, scholarships for summer study, a commitment toward district professionalism, and a genuine concern for the administrative, supervisory, and teaching staff are all factors that each school board must consider if it is to provide a modern mathematics program for the youth it serves.

Part Three

A STUDY OF NEW PROGRAMS AND SUPPLEMENTARY MATERIALS

The Subcommittee on New Programs and Supplementary Materials studied new programs that were being conducted by schools in various parts of the country along with those that were being conducted in California.

Selected Programs Studied

The Subcommittee on New Programs and Supplementary Materials first collected information concerning current programs for the improvement of elementary school mathematics in various parts of the country. Some of the experimental or study programs investigated were (1) the School Mathematics Study Group; (2) the Greater Cleveland Mathematics Program; (3) the University of Illinois Committee on School Mathematics; (4) the University of Maryland Mathematics Project; (5) the Madison Project; and (6) the Stanford University Arithmetic Program. The information gained from studying these programs aided the Advisory Committee in preparing its reports. A similar procedure should prove helpful to individuals and groups who are planning, developing, and in other ways working with mathematics programs. Certain of the information needed will be found in the *Mathematics Teacher* and the *Arithmetic Teacher*.

Programs Studied in California Public Schools

A second phase of the Subcommittee's study was to obtain first-hand information of the newer programs by directly observing them in action. The schools visited were selected by reviewing the information collected in a questionnaire study that was made by the Bureau of Elementary Education. The Cuisenaire program, Catherine Stern's Structural Arithmetic program, the Numberaid program, and others that had been developed locally were among those seen in action.

While the Subcommittee made no attempt to evaluate the various programs it observed nor the materials it saw being used, it believes the following generalizations are warranted by the observations made:

- The persons working in the programs thought mathematics to be a basic subject that merited major importance in the elementary school curriculum.

- There were variations among the programs. In some schools, textbooks from the national experimental programs were being used, some were using primarily supplementary materials, and some were using manipulative items. A few were using supplementary materials that were prepared locally, many were using some locally prepared materials in combination with other materials. Individual differences among learners were being provided for by grouping, by pacing, and by using differentiated materials and methods.
- The scope of most of the programs was not sufficiently inclusive. The programs had been in operation only a year or two and generally in only a few classes. The most extensive project observed involved 5,600 pupils and 470 teachers in 24 school districts. However, even in this program, only the upper third of the pupils in grades three through seven were receiving the instruction.
- Provision for inservice education of teachers was common. This was generally provided by consultants from colleges or the experimental programs, offices of county superintendents of schools, school district personnel, or through college courses.
- Some school districts had programs for informing parents regarding the mathematics program. In some instances parents were involved in planning the programs. The procedures being emphasized were apparently securing the desired results.
- The schools had made and were making little use of objective evaluation. However, the evidence available supported the conclusion that the programs were effective. In general, the teachers thought that the materials they were using were appropriate and well-planned. A majority of the teachers believed that their pupils were interested and were progressing well. None of the programs was considered flawless.
- Most of the districts employing the newer programs in specified grades were planning to extend them to other grades.

Subcommittee Comments

It would be hazardous to draw conclusions regarding the effectiveness of any of the programs from information as limited as that obtained by the Subcommittee. However, it may be noted that much worthwhile content is being presented in the programs and that the attitudes of the persons involved ranged from that of mild interest to real enthusiasm. Overall there was evidence of growing interest. In general, the participating pupils seemed to respond well to the content and method employed. Probably the important considerations that merit attention before any worthwhile conclusions are drawn regarding the programs would best be brought out by answers to the following questions: Was it the appeal of the new that made the participants respond as they did? Was it due to satisfaction gained from an activity currently receiving much public attention? Was the success of the programs due to the selection of teachers with special knowledge of or interest in mathematics? Was it because of the inherent satisfaction gained by both teachers and children because they found the content and manner of presentation challenging and understandable? Time and further scientific studies are needed to obtain more objective answers to these questions and others of similar type. It would seem obvious that not all of the programs are of equal merit. In the absence of comparative evaluations of a scientific nature, school districts should be cautioned to use discretion in evaluating the evidence and claims of the various programs.

Supplementary Materials

The Subcommittee on New Programs and Supplementary Materials studied available materials of a supplementary nature that might be used to implement the basic strands recommendations. The Advisory Committee on Mathematics made an earlier recommendation that the State Curriculum Commission explore the possibility of recommending the provision of supplementary mathematics materials to the public schools through a state adoption for the interim before new state textbooks are available to the schools. Such materials would be used to make the transition from traditional arithmetic to a more contemporary program.

When it became apparent that this recommendation could not be carried out, the Commission and the State Department of Education agreed that an annotated bibliography should be sent immediately to the county superintendents of schools and district superintendents of schools with the recommendation that supplementary materials should be purchased if funds were available. An annotated list of such materials was distributed during the summer of 1962.

To obtain materials for study by the Subcommittee, letters were sent by the State Department of Education to all known publishers of instructional materials in arithmetic for elementary schools requesting information concerning supplementary publications that were available or would be available at an early date. The State Department of Education then supplied members of the Subcommittee with copies of the available materials for evaluation.

In compiling the annotated bibliography, the Subcommittee defined supplementary materials as printed materials which extend the traditional curriculum in arithmetic by providing opportunity for development of the basic understandings proposed in the "Strands" statement. It agreed that the materials should contribute to a smooth transition from a traditional to a modern curriculum by both children and teachers.

In deciding which supplementary materials should be included in the bibliography, the subcommittee considered the following factors:

- Inclusion of basic concepts, as these are identified in the "Strands" statement
- Soundness of mathematical development
- Appropriateness for use at the intended level and flexibility for use at several levels
- Methods of presenting concepts
- Feasibility for classroom use
- Aids for teachers

In selecting supplementary mathematics materials from the list of materials recently available, it is recommended that the following items, as well as the preceding ones, be considered:

- The materials fit into the program.
- The materials are instructive for teachers as well as pupils and require only reasonable teacher retraining to use.
- The materials are in keeping with the basic "Strands" recommendations.

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