A Critique of the Structure of U.S. Elementary School Mathematics

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Research in mathematics education can be partitioned in many ways. If research in elementary mathematics education is partitioned into just two categories—content and method—this article belongs to the latter. It argues that consideration of the way in which the content of elementary mathematics is organized and presented is worthwhile, for both U.S. and Chinese elementary mathematics educators.

Two distinguishing features of organizing structures for elementary mathematics are categorizations of elementary mathematics content and the nature of the relationship among the categories. These features are illustrated by the two examples in Figure 1:

A. organizing structure for elementary mathematics content in China before 2001[1];

B. organizing structure in *Principles and Standards for School Mathematics*, published in 2000 by the National Council of Teachers of Mathematics (NCTM).

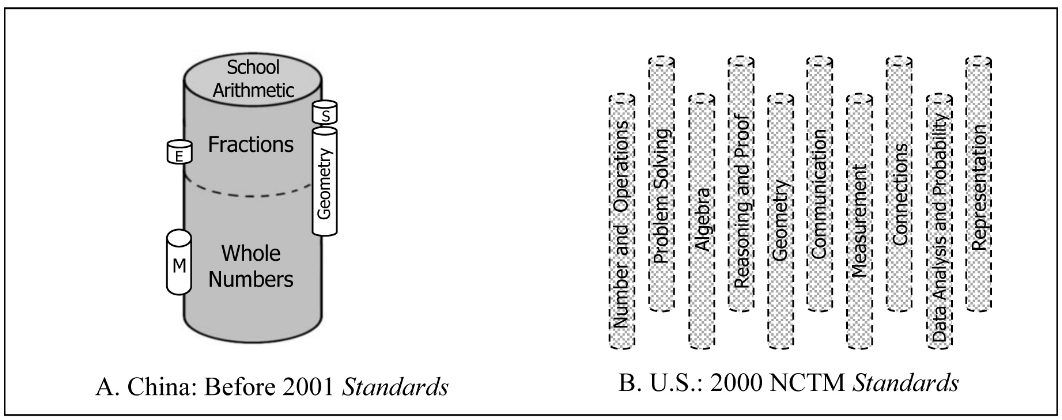

Figure 1. Two organizations of elementary school mathematics.

Example A has a "core-subject structure." The large gray cylinder in the center represents school arithmetic. Its solid outline indicates that it is a "self-contained subject." (The next section of this article elaborates the meaning of this term.) School arithmetic consists of two parts: whole numbers (non-negative integers) and fractions (non-negative rational numbers). Knowledge of whole numbers is the foundation upon which knowledge of fractions is built. The smaller cylinders represent the four other components of elementary mathematics, shown according to the order in which they appear in instruction. These are: measurement (M), elementary geometry, simple equations (E), and simple statistics (S). (The last is similar to the U.S. "measurement and

This is an English version, adapted for a U.S. audience, of an article published in Chinese in 2012 in *Journal of Mathematics Education*, *21*(4), 1–15.

[1] The reason for showing the pre-2001 organizing structure for elementary mathematics in China will be discussed later in this article.

data," and includes tables, pie charts, line graphs, and bar graphs.) The dotted outlines and interiors of these components indicate that they are not self-contained subjects. The sizes of the five cylinders reflect their relative proportions within elementary mathematics and their positions reflect their relationship with arithmetic: arithmetic is the main body of elementary mathematics, and the other components depend on it. Each non-arithmetic component occurs at a stage in the development of school arithmetic that allows the five components to interlock to form a unified whole.

Example B has a "strands structure." Its components are juxtaposed, but not connected. Each of the ten cylinders represents one standard in *Principles and Standards for School Mathematics.* The content standards appear in the front, and the process standards appear in the back. No self-contained subject is shown. This type of structure has existed in the U.S. for almost fifty years, since the beginning of the 1960s. Over the decades, the strands have been given different names (e.g., "strands," "content areas," or "standards") and their number, form, and content have varied many times.

In these two structural types, the main difference is that the core-subject structure has a self-contained subject that continues from beginning to end. In contrast, the strands structure does not and all its components continue from beginning to end of elementary mathematics instruction. School arithmetic, the core subject in Figure 1A, does not appear as a category in Figure 1B.

U.S. elementary mathematics used to have a structural type like that of Example A. However, in the 1960s it began to change radically, eventually acquiring the structural type illustrated by Example B. The next two sections describe the features, origins, and evolution of these two structural types.

"Core-subject Structure": Features, Origin, and Evolution

Notable Features of Example A

Because I was not able to obtain the pre-2001 Chinese mathematics education framework, the details shown in Figure 2 are drawn from a set of Chinese elementary textbooks published in 1988.[2] The main part of Figure 2 is school arithmetic: the content of the gray cylinder in Figure 1A. The large rectangle shows arithmetic instruction beginning at the bottom in grade 1 and continuing upward to grade 6. Light gray indicates whole number content and dark gray indicates fraction content (including decimals, ratio, and proportion).

The small boxes at the right represent the remaining four components: measurement (M), elementary geometry (G), simple equations (E), simple statistics (S). Their placement indicates their order in instruction. The arrows indicate when each non-arithmetic section occurs relative to arithmetic instruction.

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[2] In 1988, China had several series of elementary mathematics textbooks. This series was one of the most widely used and was produced by writing groups from Beijing, Tianjin, Shanghai, and Zhejiang. The organizing structure discussed in this section was common to all the textbook series used in China.

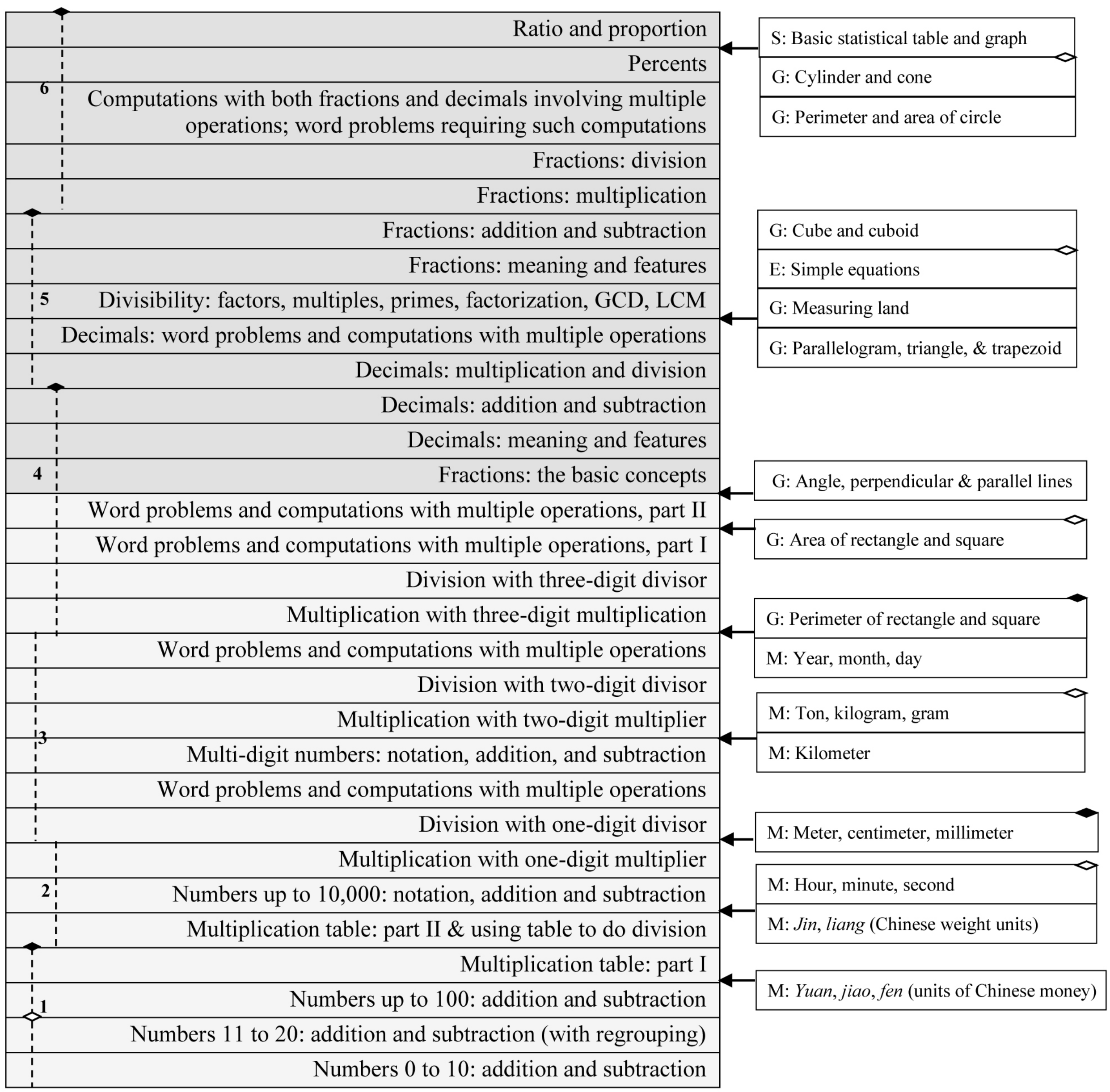

Figure 2. A pre-2001 organization of school arithmetic in China.

The dotted vertical lines at the left and numbers beside them indicate the grades in which the topics occur. The white rhombus indicates the end of the first semester, the black rhombus indicates the end of the school year.[3]

Next, we will discuss the features of Example A that are visible in Figure 2.

The first feature is the large portion of the figure occupied by arithmetic. Together, the twelve textbooks used for grades 1 to 6 have 1352 pages.[4] Arithmetic occupies 1103 pages, which is

[3] The school year in China has two semesters, each about 20 weeks long.

81.6% of the total. As for other components, measurement occupies 36 pages, elementary geometry 135 pages, simple equations 23 pages, simple statistics 18 pages, and there are 37 pages for abacus.[5] All the non-arithmetic content (including abacus) is only 18.4%.

The second feature visible in Figure 2 is the relationship between arithmetic and non-arithmetic content. Please note the insertion points for non-arithmetic content indicated by arrows. We can perceive the 30 sections of arithmetic in Figure 2 as several larger chunks, each with its own mathematical unity. That unity is supported by instructional continuity; that is, within a chunk, consecutive sections of arithmetic content occur in instruction without interruption from non-arithmetic content. For example, the first 30 weeks of instruction consist only of arithmetic—numbers 0 to 10 and their addition and subtraction, followed by numbers from 11 to 20 and their addition and subtraction (including regrouping), followed by numbers to 100 and their addition and subtraction. These three sections are tightly connected, supporting students' learning of numbers less than 100 and their addition and subtraction, thus laying a solid cornerstone for later learning. Another unified chunk is formed by the section on divisibility to the section on percents, allowing students to learn the four operations with fractions (which is difficult) without interruption. Moreover, in the twelve semesters of the six years, ten semesters start with arithmetic and non-arithmetic content occurs at the end of the semester. In this organization, arithmetic is noticeably emphasized.

The third feature visible in Figure 2 is the ordering of the instructional sections. This order attends to both mathematical relationships among calculation techniques and considerations of learning. For example, the first three sections of grade 1 are ordered by calculation technique. If technique were the sole consideration, these would be immediately followed by addition and subtraction of numbers less than 1000. However, in the textbook, the fourth and fifth sections are on multiplication tables and using multiplication tables to do division. Learning multiplication tables and doing division with them allows students to continue their study of numbers within 100 with a new approach. That is very beneficial for creating a solid foundation for elementary mathematics learning. Another example: after the section on "fractions: the basic concepts," the textbook does not immediately continue with "fractions and operations." Instead it has a section on decimals. Calculation techniques for operations with decimals are very similar to those of whole numbers, but the concept of decimals is a special case of the concept of fractions. This arrangement affords understanding of the concept of decimal, review of the four operations with whole numbers, and preparation for future learning of fractions, their properties, and operations with fractions. (Note that this organization affords but does not guarantee this understanding. Curriculum design and instruction need also to be consistent with this goal.)

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[4] Please notice the differences between Chinese and U.S. elementary mathematics textbooks. The series of textbooks for 6 years has 12 small books, one book for each semester. All their content is considered essential rather than optional. On average, each book in the series has 113 pages, with page dimensions of 8 inches by 6 inches. Of all these books, only the first uses color and that occurs on only two pages. Each student gets his or her own set of textbooks. In contrast, in the U.S. an elementary mathematics textbook for one year often has about 600 pages, frequently uses color and has pages approximately twice the size of the Chinese textbook pages. In general, the textbooks are the property of the school district than the student and students are not able to bring the textbooks home.

[5] There are two sections devoted to abacus instruction. The first, on addition and subtraction, occurs during the last unit of the first semester of grade 4. The second, on multiplication and division, occurs during the last unit of the first semester of grade 4. In order to simplify Figure 2, these are not shown.

There are also three features worth noting about the non-arithmetic sections shown in Figure 2. First, in the whole process of elementary mathematics learning, the non-arithmetic content is supported by the arithmetic content that precedes it, but at the same time, reinforces that arithmetic content. In general, each section of non-arithmetic content occurs when arithmetic learning has arrived at a stage that prepares students to learn that content. For example, the section on units of Chinese money occurs immediately after "Numbers up to 100: addition and subtraction." At that point, students have acquired significant knowledge of numbers within 100 and are able to add and subtract these numbers. That forms the foundation for students to learn the units of Chinese money. (Chinese money has three units: *fen*, *yuan*, *jiao*; 1 *yuan* is 10 *jiao*, 1 *jiao* is 10 *fen*, thus 1 *yuan* is 100 *fen*.) At the same time, learning the units of Chinese money provides students a new perspective on the arithmetic that they have just learned, allowing them to review and consolidate their prior learning. Similarly, because reading an analogue clock relies on multiples of 5, the section on units of time allows students the opportunity to apply the multiplication that they have just learned and reinforce this knowledge.

The four non-arithmetic components appear consecutively, except for one short overlap. The first component is measurement, which consists of seven instructional sections, all of which occur before third grade. Next is elementary geometry, which is formed by eight sections, distributed from third grade to sixth grade. At the end of the first semester of fifth grade, simple equations occurs. This component has only one section, and occurs between two sections of elementary geometry. After geometry, almost at the end of sixth grade, simple statistics occurs. This kind of arrangement ensures that one type of non-arithmetic content is finished before a new type begins.

The third notable feature is that the sizes of the non-arithmetic components are different. If we consider the total non-arithmetic content as 100%, their sizes are, from greatest to smallest: elementary geometry (64%), measurement (17%), simple equations (11%), simple statistics (8%). That means the core-subject structure doesn't treat non-arithmetic components equally, but emphasizes some more than others. The one receiving most emphasis is elementary geometry. In fact, if the measurement of length in the measurement component is counted as part of elementary geometry, then elementary geometry occupies even more than 64%. This noticeable emphasis on elementary geometry is associated with the mathematical content of middle school. In elementary school, arithmetic prepares the foundation for learning algebra and elementary geometry prepares the foundation for learning geometry.

In summary, if considered individually, the sections shown in Figure 2 may not seem remarkable or interesting. However, when their interrelationships are considered, these sections are revealed as a tightly connected, carefully designed six-year-long path for learning mathematics.

Essential Feature of Core-subject Structure: A Definition System

If Chinese elementary mathematics had only the features visible in Figure 2, it would not deserve the label core-subject structure. These features do not indicate whether the school arithmetic shown is a collection of skills or a self-contained subject with principles similar to those of the discipline of mathematics. The latter is true: There is a theory of school mathematics that underlies the topics shown in Figure 2.

The precursor of school arithmetic was "commercial mathematics," which existed in Europe for several hundred years after Arabic numbers were introduced. Its content was computational algorithms without explanations.[6] In the mid-nineteenth century, with the movement toward public education in the U.S. and Europe, mathematical scholars participated in producing elementary school arithmetic textbooks.[7] Their exemplar was Euclid's *Elements*, the most influential mathematics textbook in history. These scholars followed the approach of the *Elements*, striving to establish a system of definitions for operations with whole numbers and fractions. Near the end of the nineteenth century, this system was almost complete. Interestingly, although this system is fairly complete, China did not contribute to its construction. On the contrary, in the U.S. elementary mathematics textbooks of the late nineteenth century, we see the efforts and contributions of U.S. scholars.

Rather than starting from self-evident geometrical concepts such as line and point as Euclid did, these scholars began with the self-evident concept of unit to create a definition system for all of school arithmetic. For example, a number is considered a collection of units and the sum of two numbers is defined as the number which contains as many units as the numbers taken together. Subtraction is defined as the inverse operation of addition.[8] With this definition system, school arithmetic is self-contained in the sense that each of its concepts is defined in terms of previous concepts, tracing back to the starting point: unit 1. The concepts in the system are just sufficient to explain the algorithms for the four operations on whole numbers and fractions that elementary students learn. With this definition system, we can not only make coherent explanations for operations with whole numbers and fractions but can also analyze fairly complicated quantitative relationships, using the definitions of the system. For example, the problem given on the tomb of

[6] As Keith Devlin points out in Chapter 4 of his book *The Man of Numbers*, although Fibonacci provided explanations based on the *Elements* in his *Liber Abaci*, "how-to" books for commercial arithmetic focused on worked examples.

[7] The term "mathematical scholars" is intended to suggest the difference between present-day mathematicians and some of those nineteenth-century contributors. The latter included professors of mathematics such as Charles Davies, whose mathematical activity was not centered on research.

[8] See the supplementary notes at the end of this article for more details.

the famous mathematician Diophantus can be solved using such an analysis.[9] The elementary students who learn operations with whole numbers and fractions with this definition system, master the algorithms for computation while learning abstract thinking. The definition system of school arithmetic is the main part of the theory of school arithmetic. In China and some other countries, this system of definitions still underlies instruction for operations with whole numbers and fractions.[10]

Judging Whether a Country's School Arithmetic Has an Underlying System of Definitions

How can one judge whether a country has a definition system underlying its school arithmetic? The following problem can provide an efficient test. A few years ago, the field of U.S. mathematics education experienced a small shock from a word problem in a fifth grade Singapore textbook.

> Mrs. Chen made some tarts. She sold 3/5 of them in the morning and 1/4 of the remainder in the afternoon. If she sold 200 more tarts in the morning than in the afternoon, how many tarts did she make?[11]

Although people educated in the U.S. could solve this problem with non-arithmetic approaches, no one knew how to solve it using an arithmetic equation, such as:

$$200 \div [3/5 - 1/4 \times (1 - 3/5)] = 200 \div 1/2 = 400, \text{ 400 tarts}$$

Because concepts other than "unit" in the definition system are defined in terms of earlier concepts, the later a concept is defined, the more previously defined concepts it may rely on. The concept needed to solve the tarts problem occurs in the last section of the definition system. People whose elementary mathematics instruction included all the concepts of the definition system are prepared to solve this problem. Otherwise, they are not prepared to solve this problem using an arithmetic equation. Thus, testing elementary students after they learn fractions by asking them to solve this word problem provides evidence of whether or not their country still uses a fairly complete definition system.[12] This definition system underlies elementary mathematics in Singapore, thus at the end of elementary school, Singapore students are prepared

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[9] This problem is:

> God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.

Using the definition system, a solution is obtained from： $(5 + 4) \div (1/2 - 1/6 - 1/12 - 1/7) = 84$.

[10] There is no evidence that the definition system in Chinese elementary mathematics was adopted directly from the U.S.

[11] *Primary Mathematics 5A Workbook* (3rd Ed.), 1999, Curriculum Planning & Development Division, Ministry of Education, Singapore, p. 70.

[12] The time at which students are asked this question, just after learning division by fractions, is very important. Prior to that, students have not learned the entire definition system. Later, students will learn to solve such problems using algebra.

to solve this problem using arithmetic. In the U.S., not only elementary students, but their teachers and the educators of those teachers are not able to solve the problem with arithmetic. This suggests that the definition system has, at least, decayed in the U.S.

The Construction of the Subject of School Arithmetic

Examination of U.S. elementary textbooks suggests that, after the end of the nineteenth century, when most of the definition system was established, the development of school arithmetic as a subject came to a halt.[13] However, in some other countries, the development of school arithmetic continued. From the content of Chinese elementary mathematics we can see three kinds of later evolution.

First, introduction of the basic laws: commutative, associative, distributive, and compensation laws.[14] These do not appear in nineteenth-century U.S. arithmetic textbooks. With these laws, explanations for computational algorithms become more concise and applications of the algorithms become more flexible. Knowledge of these laws also forms a good foundation for learning algebra in middle school. This group of basic laws together with the definition system forms a theory of school arithmetic.

Second, types of word problems were introduced that described quantitative relationships in contexts, such as the relationship of distance, time, and velocity. These problem types came from ancient civilizations, such as Rome and China, and reflected an approach to mathematics different from that of Euclid.[15] The theory of school arithmetic established by following the approach of the *Elements* emphasizes rigorous reasoning, but these word problems provide prototypical examples of quantitative relationships. Solving variants of these problems promotes flexible use of these relationships. These types of word problem rarely appeared in late nineteenth-century U.S. elementary textbooks, but elementary textbooks in China, Russia, and perhaps other countries introduce them systematically. To solve these word problems by analyzing the quantitative relationships on the reasoning within the theory is a supplement and a contrast. The supplement comes from seeing a different approach to mathematics. The contrast comes from the differences in the two approaches and underlying mathematical traditions. Moreover, it broadens students' thinking and enriches the mathematical content.

Third, the pedagogy of school arithmetic developed further. The definition system of school arithmetic that we see in the late nineteenth-century U.S. textbooks was presented with rigorous

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[13] In some early twentieth-century writings in U.S. mathematics education such as Buckingham's *Elementary Arithmetic: Its Meaning and Practice*, we see evidence of further exploration of the definition system. However, I have found no evidence that such explorations affected elementary mathematics textbooks.

[14] Like the definition system, the basic laws are presented with words, numbers and the signs introduced in the definition system, with no symbols beyond arithmetic (letters are not used). For example, the commutative law for addition may be presented as "If two addends interchange their place, the sum does not change," with an example like "3 + 2 = 5, 2 + 3 = 5." The compensation law for addition may be presented as "If one addend increases by an amount and the other addend decreases by the same amount, the sum does not change," with an example like "5 + 4 = 9, (5 + 2) + (4 − 2) = 9." This feature of not using language beyond arithmetic is significant in terms of ensuring that the students' learning task does not go beyond their intellectual readiness.

[15] See the supplementary notes at the end of this article for more details.

wording and in logical order. Although it was rigorous according to the discipline of mathematics, as presented, it was too abstract for elementary school students. That might be a reason why the construction of the theory of school arithmetic stopped in the early twentieth-century U.S. However, with many years of effort, China and some other countries found instructional approaches for teaching school arithmetic that has this underlying theory. These approaches vary in at least three ways: in the order of the content, how it is represented, and the design of exercises. “Three Approaches to One-place Addition and Subtraction” (Ma, n. d.) gives examples of how these three aspects can work together to introduce concepts defined within a system and basic laws to first grade students.

After these changes—introduction of basic laws, introduction of prototypical problems, and pedagogical advances—the construction of school arithmetic as a subject was basically complete. It consisted of four components:

- Arabic numerals and notation for whole numbers, fractions, and operations on them, inherited from commercial arithmetic.
- Definition system augmented by basic laws.
- Prototypical word problems with variants.
- Instructional approaches.

This school arithmetic was self-contained: it had an underlying theory following the approach of the *Elements*. It was open: although based on the Euclidean tradition of Greek mathematics, it included mathematical traditions of other civilizations. It was teachable: developments in pedagogical approaches created a school arithmetic that was learnable by following the character of students’ thinking and leading students step by step to progressively more abstract thinking.

In the U.S., although much of the definition system had been established by the end of the nineteenth century, during several decades of the progressive education movement, the three types of developments discussed above did not occur in significant ways. Thus, in the U.S., the construction of school arithmetic as a subject was never really completed. In other words, a well developed school arithmetic never really existed as a subject to be taught to U.S. children.

U.S. Strands Structure: Origin, Features, and Development

Although it never had a well developed school arithmetic, after the beginning of elementary education, arithmetic was the core of elementary mathematics in the U.S. for almost 100 years. However, today’s U.S. elementary mathematics has a different type of structure. When did this change happen? How did it evolve into the structure shown in Figure 1B? What did this change in structure mean to U.S. elementary mathematics? We begin with the creation of this structure in the first California mathematics framework.

During the past few decades, the relationship between mathematics education in California and the rest of the nation has been intriguing. In some sense, we can say that California has been the forerunner of the rest of the United States. Suzanne Wilson, in her book *California Dreaming: Reforming Mathematics Education* mentions several times how mathematics reform in California has influenced that of the entire nation. She pointed out that the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics* and 1985 *California Mathematics Framework* "drew on the same research, commitments, and ideas" (2003, p. 26). What we will discuss here, however, is an even earlier, more profound California influence on national mathematics education. This is the fundamental change in the structure of elementary mathematics content initiated by the first California mathematics framework known as "The Strands Report."

The First California Mathematics Framework: Creation of Strands Structure

In October of 1957 the Soviet Union launched Sputnik. This unusual event caused the U.S. to reflect on its science and mathematics education. In 1958, the National Science Foundation funded the School Mathematics Study Group (SMSG) led by the mathematician Edward Begle to promote the reform of U.S. mathematics curriculum, later known as the "new math."

In 1960, California formed the State Advisory Committee on Mathematics with Begle as its chief consultant to launch the statewide reform of mathematics education. The committee was composed of three subcommittees. The first subcommittee on "Strands of Mathematical Ideas" consisted mainly of mathematics professors. Its charge was to decide the new mathematical structure of the new curriculum. The charge of the other two subcommittees was to implement the new curriculum, one to decide how to prepare teachers for the new curriculum, and the other to investigate "the more recent 'new' mathematics programs that have attracted national attention" and study "commercially produced materials that could be used profitably to supplement the state adopted materials" (p. v).

In 1963, the reports from the three committees were published as *Summary of the Report of the Advisory Committee on Mathematics to the California State Curriculum Commission.* Because its main section was "The Strands of Mathematics," the whole report was known as the "Strands Report." Later, its official name became the *First California Mathematics Framework*. In this report, the arithmetic-centered structure of elementary mathematics was replaced by a different type of structure consisting of juxtaposed components—creating the strands structure.

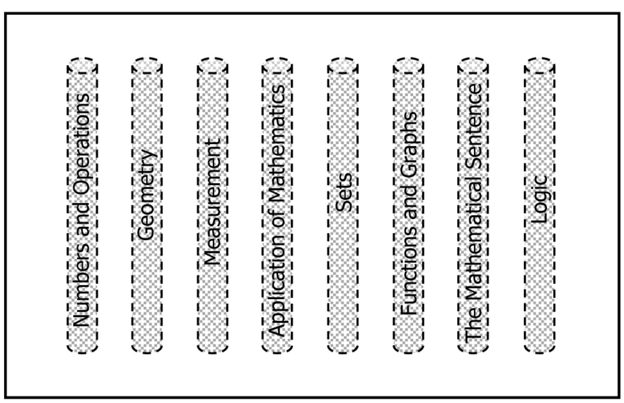

Figure 3. The structure of elementary school mathematics suggested by the first Strands Report.

The Strands Report began:

> The curriculum which we recommend departs but little from the topics normally studied in kindergarten and grades one through eight, topics which long ago proved their enduring usefulness. But it is essential that this curriculum be presented as one indivisible whole in which the many skills and techniques which compose the present curriculum are tied together by *a few basic strands of fundamental concepts* which run through the entire curriculum. (pp. 1–2, emphasis added)

According to the report, these "basic strands" are:[16]

1. Numbers and operations
2. Geometry
3. Measurement
4. Application of mathematics
5. Sets
6. Functions and graphs
7. The mathematical sentence
8. Logic

The idea of tying together elementary mathematics content with these eight strands meant a two-fold revolution in the elementary mathematics curriculum. One was the revolution in the components of content. Arithmetic was no longer considered to be the main content. Instead, concepts from advanced mathematics such as sets, functions, and logic were introduced into elementary mathematics. Second was a revolution in the structural type, establishing a new strands structure for elementary mathematics.

Each of the eight strands was represented as a few concepts from a branch of mathematics, but not as a self-contained subject. Although the report discussed a few important concepts for each strand, there was no evidence that these important concepts sufficed to form a system that provided explanations for the operations of elementary mathematics. For example, the strand "Numbers and operations," which might be considered closest to arithmetic, included fifteen concepts, such as one-to-one correspondence, place value, number and numeral, and Cartesian product (CSED, pp. 4–13).[17] The report claimed that these fifteen concepts were important but did not explain how they were related. The other seven strands were discussed in a similar way.

From the earlier quotation, we can see that the authors of the Strands Report intended to express all of elementary mathematics in terms of a few basic concepts and thus unify its content. However, realization of this idea was not widespread in U.S. elementary education. In this way,

[16] The report may have drawn on two intellectual resources: Bruner's *Process of Education* (1960) and Nicolas Bourbaki's *Elements of Mathematics*. Discussion of their possible influence on the first Strands report is beyond the scope of this article.

[17] The 15 concepts in the Numbers and Operations strand are: One-to-one correspondence (p. 4), Place value (p. 5), Number and numeral (p. 6), Order: The number line (p. 6), Operations: Cartesian-product (pp. 7–8), Array (p. 7), Closure (p. 8), Commutativity (p. 9), Associativity (p. 9), Identity elements—zero and one (p. 10), Distributivity (p. 10), Base (p. 11), The decimal system (p. 12), Square root (p. 13).

the newly introduced concepts from advanced mathematics did not unify elementary mathematics although the earlier definition system of school arithmetic was officially abandoned. Since then, the concepts in U.S. elementary mathematics education have never had an underlying definition system that played the same role as the earlier one.

Today, when we read the Strands Report, we should admit that it has some interesting and inspiring ideas and discussions. We can also understand that its authors, facing concerns about national security at that time, wished to introduce concepts from advanced mathematics. However, maybe because of insufficient time or other reasons that we don't know, they didn't even make an argument for the new approach. Why change the previous elementary mathematics curriculum with arithmetic at its center into this curriculum with eight strands?[18] What is the advantage of doing so? Why these eight strands and not others? Why were these eight strands the most appropriate for making elementary mathematics into "one indivisible whole"?

The Strands Report presented the wish of unifying elementary mathematics curriculum into an indivisible whole, however, its structure militated against this aim. It did not present evidence that the eight strands would form an indivisible whole. Moreover, because its structure consisted of juxtaposed categories without evidence to show that these categories were the only appropriate choice, changes in the strands were unavoidable. In fact, such changes occurred only a few years later, and have continued to occur.

The direct result of representing elementary mathematics as a strands structure is that arithmetic stopped being its core. The Strands Report put some "arithmetic topics" into the Numbers and Operations strand, and some into other strands. From that point on, school arithmetic, which had stopped its development in the early twentieth century, officially disintegrated.

The Second California Mathematics Framework: First Change in the Strands

Four years later, in 1967, the California State Mathematics Advisory Committee submitted its second Strands Report. This was formally published in 1972 as *The Second Strands Report: Mathematics Framework for California Public Schools.* In this second report, the number and names of the strands changed.

The "Mathematical sentence" strand was removed. Two new strands, "Statistics and probability" and "Problem solving" were added, changing the number of strands from eight to nine. At the same time, the "Logic" strand was changed to "Logical thinking." No explanation of these changes was given in the document.

[18] The report says: "The arithmetic . . . must *not* appear to the pupil as a sequence of disconnected fragments or computational tricks. Some of the important unifying ideas are discussed briefly in the following section of this report" (p. 4). The report did not mention that there was a self-contained definition system underlying late nineteenth-century U.S. elementary mathematics. From this, I infer that the authors of the report were not aware of the definition system.

The eight strands in first Strands Report	
Strand 1	Numbers and Operations
Strand 2	Geometry
Strand 3	Measurement
Strand 4	Application of Mathematics
Strand 5	Functions and Graphs
Strand 6	Sets
Strand 7	The Mathematical Sentence
Strand 8	Logic

The nine strands in second Strands Report	
Strand 1	Numbers and Operations
Strand 2	Geometry
Strand 3	Measurement
Strand 4	Applications of Mathematics
Strand 5	Statistics and Probability
Strand 6	Sets
Strand 7	Functions and Graphs
Strand 8	Logical Thinking
Strand 9	Problem Solving

Figure 4. Changes in the strands: 1963 to 1972.

As we have seen, the strands structure allowed an unlimited number of possibilities for changing the names, number, content, and features of the strands. After this, U.S. elementary mathematics lost its stability and coherence. After only four years, the same mathematics professors who wrote the first Strands Report changed the strands without explanation. In a strands structure, no strand was self-contained, moreover, the relationship among the strands was such that individual strands could be readily changed. Anyone writing a framework could easily change the content of mathematics education by changing a strand. Later, when the main authors of the mathematics framework were not mathematicians, but teachers and cognitive scientists, they retained its structure, but changed its strands to fit their views of mathematics education.

The "Back to Basics" Framework: Same Structure, Different Vision

Three years after the publication of the second Strands Report, the direction of mathematics education in California changed dramatically. The education department decided to give up the "new math" promoted in the two earlier frameworks and emphasize "the acquisition of basic mathematics skills" (CSDE, 1975, Preface). An Ad Hoc Mathematics Framework Committee was formed, led by a high school teacher. In 1975, the third California mathematics framework was published.[19] The Superintendent of Public Instruction wrote in the foreword that although this new framework could be called a "post–new math framework," he himself preferred to call it the "basics framework." He emphasized that "The contents reflect the concerns of teachers rather than those of mathematicians." It is obvious that the vision of this framework is fundamentally different from that of the earlier two. The vision of the mathematicians who wrote the first two Strands Reports was abandoned. Mathematicians were no longer the leaders in writing frameworks.

As a sign of the end of "new math," the "Sets" strand was removed. The new framework combined "Problem solving," the last strand of the previous framework, with its fourth strand "Application of mathematics." Two strands had their names changed: "Numbers and operations"

[19] From this version on, California frameworks addressed grades K through 12. In this article, I discuss only the aspects of the frameworks that pertain to elementary mathematics.

changed to “Arithmetic, numbers and operations” and “Functions and graphs” changed to “Relations and functions” (see Figure 5).

The nine strands in second California framework	
Strand 1	Numbers and Operations
Strand 2	Geometry
Strand 3	Measurement
Strand 4	Applications of Mathematics
Strand 5	Statistics and Probability
Strand 6	Sets
Strand 7	Functions and Graphs
Strand 8	Logical Thinking
Strand 9	Problem Solving

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The seven strands in “Basics Framework”	
Strand 1	Arithmetic, Numbers and Operations
Strand 2	Geometry
Strand 3	Measurement
Strand 4	Problem Solving and Applications
Strand 5	Probability and Statistics
Strand 6	Relations and Functions
Strand 7	Logical Thinking

Figure 5. Changes in the strands: 1972 to 1975.

Another type of change in the “basics framework” was the way in which objectives were presented. The first framework had abandoned traditional presentation of content by grade, instead combining the content of grades K–8.[20] The basics framework used a different arrangement and presented the K–8 objectives for each strand by grade bands: K–3, 4–6, 7–8 (see its Appendix A). A third type of change was the creation of two kinds of strand categories: “strands” and “content areas” (1975, CSDE, p. 11).

Between the “new math” and “back to basics” eras, the vision of mathematics education changed fundamentally. However, the structure created by the mathematicians who wrote the first framework remained. This structure allows fundamental changes in vision to be presented simply by changing the strand categories.

1980 Addendum to the Framework: Problem Solving Above All

A few years after the “basics framework” was published, the California State Department of Education published *Mathematics Framework and the 1980 Addendum for California Public Schools, Kindergarten through Grade Twelve,* consisting of the “basics framework” together with an addendum. The number of pages in the *Addendum* was four-fifths the number of pages in the *Framework*, in order to revise the vision of the “basics framework”:

> As the result of the back-to-basics movement, there is a tendency by some educators to allocate too much time in mathematics classes to working with drill and factual recall. With the advent of low cost, high performance microprocessors

[20] The first Strands report said:

> Definite grade placement is not crucial. We must recognize that concepts are not mastered immediately but are built up over a period of time. It is therefore important that these notions be introduced as early as possible and be linked with the pupils’ mode of thinking. Above all we seek, to the limit of each pupil’s capability, his understanding of that unified mathematical structure which is the content of K–8 mathematical study. (CSDE, 1963, p. 2)

and calculators, it becomes possible for computations to be done more accurately and in less time than in the past. This allows more time for problem solving, the major focus of the mathematics curriculum. (CSDE, 1982, p. 75)

To emphasize the importance of problem solving, the addendum took one strand from the previous framework and made it an umbrella for the other strands, which were called "skill and concept areas" to emphasize their subordinate role (CSDE, 1982, p. 59). This idea was emphasized by a figure (see Figure 6).

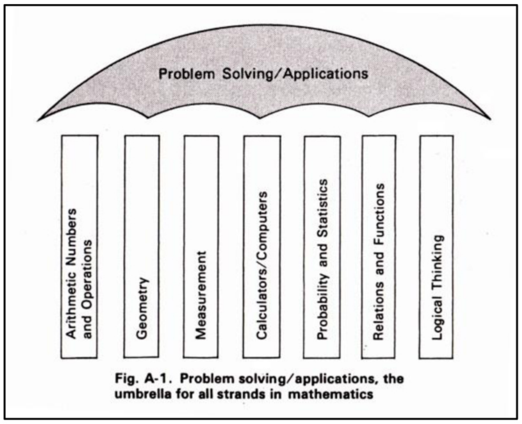

Figure 6. The 1980 addendum emphasized problem solving (CSDE, 1982, p. 60).

The approach of the 1980 addendum differentiated the strands, making some more important than others. This approach was continued in the next framework and influenced the 1989 NCTM *Curriculum and Evaluation Standards*.

1985 *California Framework* and 1989 NCTM *Curriculum and Evaluation Standards*: Creation of Subitems

The 1985 California framework and the 1989 NCTM standards shared a similar new vision of mathematics education. The new, exciting vision presented in these two documents was to let every student, not only academic elites, acquire "mathematical power" (CSDE, 1985) and become "mathematically literate" (NCTM, 1989).

As mentioned earlier, during the new math movement mathematicians intended to use fundamental mathematical concepts such as "set" and "function" to explain the content of elementary mathematics. However, the 1980s round of reform seems to have been influenced by cognitive science. Terms related to cognition, such as "ability," "cognition," "number sense," "spatial sense," "to communicate," "to understand," appear frequently in these documents.

The 1985 California framework stated:

> Mathematical power, which involves the ability to discern mathematical relationships, reason logically, and use mathematical techniques effectively, must be the central

> concern of mathematics education and must be the context in which skills are developed. . . . The goal of this framework is to structure mathematics education so that students experience the enjoyment and fascination of mathematics as they gain mathematical power. (pp. 1–2)

The 1989 NCTM standards suggested that to become mathematically literate involved five goals for students:

1. to value mathematics
2. become confident in their ability to do mathematics
3. become mathematical problem solvers
4. to learn communicate mathematically
5. to learn to reason mathematically.

The authors were convinced that if students were "exposed to the kinds of experience outlined in the *Standards*, they will gain *mathematical power*" (p. 5).

The structural type of the NCTM standards was visibly influenced by the earlier California frameworks. Although the frameworks referred to "strands" or "areas" and the standards referred to "standards," these items were organized in very similar ways. Via the NCTM standards, the strands structure that originated in the first California framework had a national impact. Wilson wrote:

> The boundaries between the national and California discussions of mathematics education and its reform were porous and permeable. It was hard—as observers—to separate those discussions and to determine where ideas originated. . . . Many California schoolteachers were part of the writing of and the reaction to the development of the NCTM *1989 Standards*. (2003, p.127)

Although it was based on the 1980 addendum, the 1985 framework had a new kind of item: "major areas of emphasis that are reflected throughout the framework" (CSDE, 1985, p. 2) that occurred before the discussion of the strands. There were five of these areas:

1. problem solving
2. calculator technology
3. computational skills
4. estimation and mental arithmetic
5. computers in mathematics education.

The 1985 framework changed some of the strands. "Problem solving/application" changed from a strand to a major area of emphasis and a new strand called "Algebra" was added. Some strands had their names changed: "Arithmetic, number and operation" to "Number"; "Relations and functions" to "Patterns and functions"; "Logical thinking" changed back to "Logic"; "Probability and statistics" to "Statistics and probability." In this way, five major "areas of emphasis" plus seven "strands" or "areas" became the twelve parts of the strands structure in the new framework.

This framework created a new kind of strands structure that had items and subitems. Under each strand, a list of student understandings and actions was given. Although they were stated as objectives, their content suggested a partition of each strand. For example, under “Number,” there are seven objectives. Under “Measurement,” there are nine objectives. The 1985 framework had seven strands, with 41 objectives.

The NCTM 1989 standards used a similar structure. It set up 13 standards:

> Each standard starts with a statement of what mathematics the curriculum should include. This is followed by a description of the student activities associated with that mathematics and a discussion that includes instructional examples. (NCTM, 1989, p. 7)

Like the 1985 framework, the 1989 NCTM standards separated the 13 standards into two groups. The first four standards formed one group shared by all the grades: “Mathematics as problem solving,” “Mathematics as communication,” “Mathematics as reasoning,” and “Mathematics as connections.” The remaining nine standards formed a group related to content and their names differed by grade band. For example, the names of the K–4 standards were “Estimation,” “Number sense and numeration,” “Concepts of whole number operations,” “Geometry and spatial sense,” “Measurement,” “Statistics and probability,” “Fractions and decimals,” and “Patterns and relationships.

Similar to the 1985 framework, the 1989 NCTM standards had several objectives listed for each standard. Thus, both the 1985 framework and the 1989 standards had two layers, items (strands or standards), each of which was followed by a bulleted list of more detailed subitems. In general, a teacher concerned about addressing the standards seems more likely to have attended to the bullets rather than the more general strands or standards. For example, when a K–4 teacher sees the first standard “Mathematics as problem solving,” he or she might think of one standard. When seeing the five bullets listed for this standard, his or her attention may be attracted to these, because they are supposed to be implemented in teaching. Thus, the content and number of subitems may have a direct impact on classroom teaching.

Table 1 lists the thirteen standards and 56 bullets of the K–4 standards (NCTM, 1989, pp. 23–61).

Table 1. 1989 NCTM K–4 Standards and Bullets

Standard 1: Mathematics as problem solving **1.** Use problem-solving approaches to investigate and understand mathematical content; **2.** Formulate problems from everyday and mathematical situations; **3.** Develop and apply strategies to solve a wide variety of problems; **4.** Verify and interpret results with respect to the original problem; **5.** Acquire confidence in using mathematics meaningfully. **Standard 2: Mathematics as communication** **6.** Relate physical materials, pictures, and diagrams to mathematical ideas; **7.** Reflect on and clarify their thinking about mathematical ideas and situations; **8.** Relate their everyday language to mathematical language and symbols; **9.** Realize that representing, discussing, reading, writing, and listening to mathematics are a vital part of learning and using mathematics. **Standard 3: Mathematics as reasoning** **10.** Draw logical conclusions about mathematics; **11.** Use models, known facts, properties, and relationships to explain their thinking; **12.** Justify their answers and solution processes; **13.** Use patterns and relationships to analyze mathematical situations; **14.** Believe that mathematics makes sense. **Standard 4: Mathematical connections** **15.** Link conceptual and procedural knowledge; **16.** Relate various representations of concepts or procedures to one answer; **17.** Recognize relationships among different topics in mathematics; **18.** Use mathematics in other curriculum areas; **19.** Use mathematics in their daily lives. **Standard 5: Estimation** **20.** Explore estimation strategies; **21.** Recognize when an estimate is appropriate; **22.** Determine the reasonableness of results; **23.** Apply estimation in working with quantities, measurement, computation, and problem solving. **Standard 6: Number sense and numeration** **24.** Construct number meanings through real-world experiences and the use of physical materials; **25.** Understand our numeration system by relating counting, grouping, and place-value concepts **26.** Develop number sense; **27.** Interpret the multiple uses of numbers encountered in their real world.	**Standard 7: Concepts of whole number operations** 28. Develop meaning for the operations by modeling and discussing a rich variety of problem situations; 29. Relate the mathematical language and symbolism of operations to problem situations and informal language; 30. Recognize that a wide variety of problem structures can be represented by a single operation; 31. Develop operations sense. **Standard 8: Whole number computation** 32. Model, explain, and develop reasonable proficiency with basic facts and algorithms; 33. Use a variety of mental computation and estimation techniques; 34. Use calculators in appropriate computational situations; 35. Select and use computation techniques appropriate to specific problems and determine whether the results are reasonable. **Standard 9: Geometry and spatial sense** 36. Describe, model, draw, and classify shapes; 37. Investigate and predict the results of combining, subdividing, and changing shapes; 38. Develop spatial sense; 39. Relate geometric ideas to number and measurement ideas; 40. Recognize and appreciate geometry in their world. **Standard 10: Measurement** 41. Understand the attributes of length, capacity, weight, mass, area, volume, time, temperature, and angle; 42. Develop the process of measure ring and concepts related to units of measurement; 43. Make and use estimates of measurement; 44. Make and use measurements in problem and everyday situation. **Standard 11: Statistics and probability** 45. Collect, organize, and describe data; 46. Construct, read, and interpret displays of dada; 47. Formulate and solve problems that involve collecting and analyzing data; 48. Explore concepts of chance. **Standard 12: Fractions and decimals** 49. Develop concepts of fraction, mixed numbers, and decimals; 50. Develop number sense for fractions and decimals; 51. Use models to relate fractions to decimals and to find equivalent fractions; 52. Use models to explore operations on fractions and decimals; 53. Apply fractions and decimals to problem situations **Standard 13: Patterns and relationships** 54. Recognize, describe, extend, and create a wide variety of patterns; 55. Represent and describe mathematical relationships; 56. Explore the use of variables and open sentences to express relationships.

If one has the patience to read all 56 bullets in Table 1, one will find that many descriptions are vague and the relationships of the bullets are not visible. Several items are hard to understand without further explanation,[21] which the document does not give.

The 1989 NCTM standards were intended to guide teachers' mathematics teaching by providing standards for curriculum and assessment. But, in my opinion, whether the reader is a curriculum designer, assessment creator, or teacher, these 56 bullets will be overwhelming and eventually ignored. Here I must point out that the strands structure opened the door for the existence of this disconnected list.

In 1992, California published a new mathematics framework in order to make the content taught in California closer to that described in the NCTM standards. These, in turn, had been inspired by the previous California framework.

A major disadvantage of representing the goals of elementary mathematics instruction in terms of cognitive actions or behaviors related to mathematics is that implementation of such descriptions is difficult. Of course mathematics learning is a cognitive activity. However, the descriptors of those cognitive activities are often vague and have multiple meanings, even for cognitive scientists. For example, "developing number sense" (bullet 26) is an important goal of the 1989 NCTM standards. However, what is number sense? There are so many interpretations, it is hard to choose. In his article "Making Sense of Number Sense," Daniel Berch noted, "Gersten et al. pointed out that no two researchers define number sense in exactly the same way. What makes this situation even more problematic, however, is that cognitive scientists and math educators define the concept of number sense in very different ways" (2005, p. 333). Then "after perusing the relevant literature in the domains of mathematical cognition, cognitive development, and mathematics education," Berch "compiled a list of presumed features of number sense" (2005, p. 334). To require teachers to work towards teaching goals which are so vaguely defined is not practical.

1999 California Framework, 2000 NCTM Standards: Establishment of Sub-subitems

In 1999, California published its sixth mathematics framework. In 2000, NCTM published its *Principles and Standards for School Mathematics*. The vision of these two documents differed from that of their three immediate predecessors. The computational skills deemphasized in the previous round of reform received more emphasis. The 1999 framework arranged mathematical content by grades and changed "strands" to "standards." The 2000 NCTM standards reduced the previous 13 standards to 10. The last five of these standards, which were similar to the first four standards of 1989, were called "process standards." The first five were called "content standards."

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[21] Some examples occur in Standard 8: Whole number computation which has as the last three of its four goals:

> Relate the mathematical language and symbolism of operations to problem situations and informal language.
>
> Recognize that a wide variety of problem structures can be represented by a single operation.
>
> Develop operations sense.

Thus, for example, in 1989, "Mathematics as problem solving" was first in the list of 13 standards. In 2000, "Problem solving" was sixth in the list of 10 standards, suggesting a change in status.

In these two documents, items and subitems appeared as in previous documents, together with a new structural feature: sub-subitems. The 1999 framework had five standards. Each standard had subitems, and each subitem had sub-subitems. The 2000 NCTM standards retained the structure of the 1989 standards, with each standard partitioned into several goals. A new structural feature, however, was that under each content standard goal were listed several expectations. For example, in the PreK–2 grade band, the first standard, "Number and operation," is partitioned into three goals. Together the three goals consist of 12 expectations. For PreK–2, the total number of goals for process standards and expectations for content standards is 63. Thus, compared with the 1989 standards, although the number of standards was reduced, there was an increase in the number of the most specific items. The continued increases in the number of specific items in the strands structure may be an important reason why the U.S. elementary mathematics curriculum became broader and shallower.

In 2006, in order to change the "mile wide and inch deep" U.S. curriculum,[22] NCTM made a significant move, publishing *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (known as "Focal Points") which provided "descriptions of the most significant mathematical concepts and skills at each grade level" (NCTM, 2006, p. 1). Focal Points adopted the approach of California's 1999 framework of arranging content by grade rather than grade band.

Focal Points gives three focal points for each year, with detailed descriptions of each. This approach helps to emphasize a few ideas, however, given that the strands structure was retained, there is no single self-contained subject which served as the core of the curriculum. The wide and shallow curriculum could become narrower because the number of items had been reduced, but narrowing does not automatically produce depth.

Conclusion

Notable Aspects of the Strands Structure: Not Stable, Not Continuous, Not Coherent

The big influence of the first California framework, the Strands Report, published during the new math movement was to fundamentally change the structure of U.S. mathematics content from a core-subject structure to a strands structure. During the past few decades, although the names of items changed from "strands" to "areas" to "standards," the strands structure remained. The damage this structure caused to U.S. elementary mathematics education is the instability of content, discontinuity in instruction, and incoherence in concepts.

[22] See NCTM, 2006, p. 3 and Schmidt et al., 1997.

The title, the content, and the number of the items forming elementary mathematics content by some people (for example, a committee) can easily be changed by a group of other people (for example, another committee). For example, the ten framework and standards documents discussed earlier almost always had different names for items. There were 94 different names for items on the first layer. Of these, 28 names were used only once and 14 names were used only twice. The most stable were "Geometry" and "Measurement." These appeared in six documents.

Instability. The instability caused by the strands structure is certainly a cause of complications for practitioners such as teachers, textbook authors, and test developers. Each change in number, names, and content of items, even changes in their order requires a response, sometimes a response significantly different from previous responses. Such changes every few years are difficult, even for experts in mathematics education, not to mention elementary teachers who teach several subjects. Such frequent changes even affect parents, making it difficult for them to tutor their children in elementary mathematics.

Another damage caused by instability is that experience cannot accumulate. For the development of any field, accumulation over time is very important. This requires the content of the field to be basically stable. If it is always changing, abandoning, or creating, it is hard for us to keep meaningful things. The content and pedagogy of core-subject elementary mathematics were developed and formed over a long period. The terms in the definition system of school arithmetic have been used for several hundred years. Following the approach of the *Elements*, which is two thousand years old, mathematical scholars revised the definitions of these terms and established a definition system. Then, based on this definition system, they established a theory of school arithmetic. After several more decades, textbook authors and teachers developed a way to teach school arithmetic based on this theory to elementary students. Part of this approach is in the textbooks, part is in journal articles, and part is in the mouths and ears of elementary teachers. This accumulation of knowledge and experience is not easily noticed, but can only occur in a relative stable situation.

Discontinuity. As mentioned before, in the core-subject structure, the continuity of instruction was protected and ensured. In the strands structure, the content of instruction needs to jump from item to item. Because of these jumps and because so many items are to be addressed at the same time, it is impossible for U.S. elementary school students' learning to have continuity. This is an invisible but severe injury to students' learning.

Incoherence among concepts. The many concepts of the present-day U.S. elementary curriculum do not cohere. Some come from current standards and some remain from earlier standards. For example, although the new math movement has been dormant for a long time, some of its concepts remain in elementary education. For the meanings of the four operations on whole numbers, and the concepts of addition and subtraction on whole numbers, the eleven models described by researchers who study children's cognitive activity are popular. Concepts of multiplication on whole numbers include "repeated addition," "equal-sized groups," and various interpretations of "Cartesian product." But, concepts of the four operations on fractions are not described in ways that are connected to these operations on whole numbers.

The field of mathematics education has noticed this incoherence. Almost all of the ten frameworks and standards documents created since the 1960s mention the idea of unification. However, this unification was never widespread in school mathematics during these decades. In my opinion, the largest obstacle to unification is the strands structure. In order to get their textbooks adopted, publishers need to demonstrate adherence to state frameworks or standards. This is often done by using their categories and organization.[23] If curriculum materials adhere to the strands structure without further unification of the concepts, then unification becomes the responsibility of teachers. Teachers who are able to do this, must: 1) have a unified and deep knowledge of the discipline of mathematics; 2) be very familiar with features of student learning. It is not impossible to produce people who meet these requirements, but it has a very high social cost. The quantity of elementary school mathematics teachers is so large that producing sufficient numbers of such people is an extremely difficult problem.

I think that readers will agree with me, that these features—instability, discontinuity of teaching and learning, and incoherence among concepts—have damaged U.S. elementary student learning. These are inherent in the strands structure.

To Reconsider School Arithmetic and Its Potential: One Suggestion for U.S. Mathematics Education

Is there any subject that can unify the main content of elementary mathematics: the four operations on whole numbers and fractions, their algorithms, and quantitative relations? Yes, this is school arithmetic. U.S. scholars contributed to this arithmetic as it was being constructed, but it was left to other countries to complete this process and make it teachable.

Since the early 1960s, from the new math until today, in U.S. elementary mathematics, I see a trend of pursuing advanced concepts, such as set theory, number theory, functions, and advanced cognitive abilities, such as problem solving, mathematical thinking, and "thinking like a mathematician." During recent decades, efforts have been made to put algebra content in early grades. It seems that only by pursuing those advanced concepts and abilities can the quality of school mathematics be raised. However, together with the intent to pursue advanced concepts and cognitive abilities, we see "math phobia" among teachers and students.

One reason that U.S. elementary mathematics pursues advanced ideas is that the potential of school arithmetic to unify elementary mathematics is not sufficiently known. This is a blind spot for current U.S. elementary mathematics. One popular, but oversimplified, version of this trend is to consider arithmetic to be to be solely "basic computational skills" and consider these basic computational skills as equivalent to an inferior cognitive activity such as rote learning. Thus, for many people arithmetic has become an ugly duckling, although in the eyes of mathematicians it is often a swan.

I would like to mention two things: 1) some countries considered to have good mathematics education such as Singapore have elementary mathematics with arithmetic as its core subject. 2) the Russian elementary mathematics textbooks with algebraic content which have attracted attention from those concerned about U.S. elementary mathematics has a underlying theory of

[23] For an example, see Ginsburg, Klein, & Starkey,1998, pp. 437–438.

school arithmetic from grade 1 onward. The algebra content in the Russian elementary textbooks which are available in the U.S. is founded on this underlying theory.

The precursor of school arithmetic was "commercial mathematics," a collection of algorithms without explanations for computing operations on whole numbers and fractions. This may not qualify as mathematics. However, after scholars built a theory in the manner of the *Elements*, commercial mathematics was reborn as a subject that embodied mathematical principles. The profound understanding of fundamental mathematics that I discussed in my book (Ma, 1999) is an understanding of this reborn arithmetic from a teacher's perspective. The Chinese elementary teachers that I interviewed for my book *Knowing and Teaching Elementary Mathematics* had not studied any advanced mathematics. However, most had a sound understanding of elementary mathematics. A subgroup of very experienced teachers (about 10% of my sample) had what I called a "profound understanding of fundamental mathematics." Their profound understanding was acquired by studying and teaching school mathematics with this arithmetic as its core.[24] What I called "fundamental mathematics" is more accurately described as the foundation for learning mathematics. School arithmetic is the cornerstone of this foundation. Therefore, I suggest that U.S. elementary mathematics education reconsider school arithmetic, its content, and its potential in mathematics education.

Chinese Curriculum Standards: A Cautionary Tale

In 2001 and 2012, China published mathematics curriculum standards (called, respectively, "experimental version" and "2011 version"). Reading these curriculum standards, we can see the authors' intent to have a Chinese character, however, it is obvious that they have made significant borrowings from the 1989 NCTM standards in ideas, wording, and writing style.

One of my main concerns, however, is that Chinese curriculum standards, like the first California framework of 1963, may radically change the structure of Chinese elementary mathematics. The 2001 Chinese standards have four categories of general goals. These four categories and four areas are very similar to the two groups of standards in the 1989 *Curriculum and Evaluation Standards*. In this way, the previous Chinese core-subject elementary mathematics has been changed to a strands structure.

[24] See Chapter 6 of Ma, 1999.

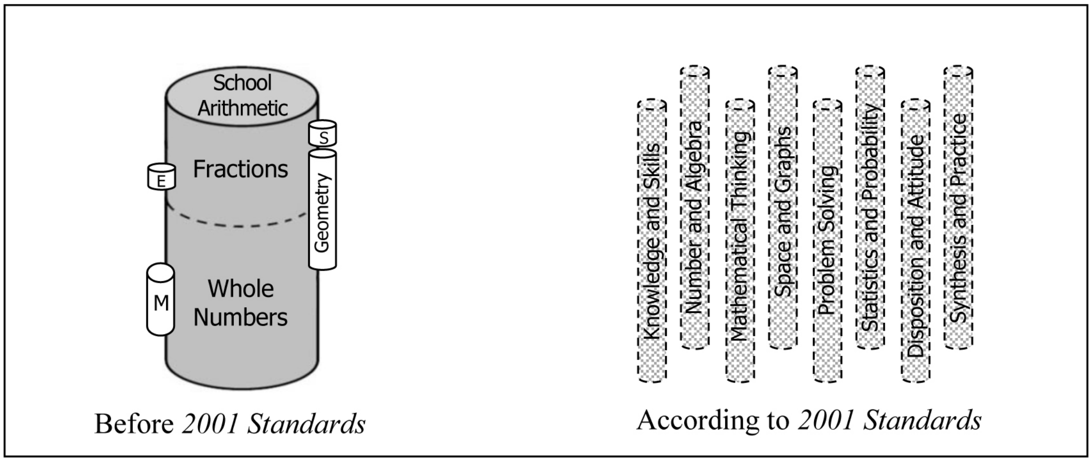

Figure 7. Changes in the structure of Chinese elementary mathematics.

The eight items have the same noticeable features as the U.S. strands items: 1) there is no item that is a self-contained subject; 2) the relationships among the items are not described. Therefore, although there are phrases such as "connect," "tightly connected," "interwoven," these phrases are used in a way that is similar to the "unify" and "consistent" discussed before, that is, they have no concrete referent. In this way, Chinese elementary mathematics may follow U.S. elementary mathematics, and step by step become unstable, inconsistent, and incoherent. In fact, the problems of the strands structure have already appeared in China.

1. Number sense
2. Symbol sense
3. Spatial concepts
4. Concept of statistics
5. Disposition to apply mathematics
6. Reasoning ability

2001 Chinese standards, p. 5

⇨

1. Number sense
2. Disposition to attend to symbols
3. Spatial concepts
4. Geometric intuition
5. Data analysis concepts
6. Computational ability
7. Reasoning ability
8. Disposition to model
9. Disposition to apply mathematics
10. Disposition for innovation

2011 Chinese standards, pp. 4–5

Figure 8. Change in Chinese standards, 2001 to 2011.

We must note that the influence of structural change will be deep and long. Visions of education can be adjusted. Teaching methods can be revised. But if the structure of the subject to be taught decays, it is hard to restore it. No matter what direction mathematics education reform takes, we should not ignore questions about changes in the structure of the subject as it is organized and presented to teachers, curriculum designers, assessment developers, and others concerned with mathematics education.

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Supplementary Notes: Definition System, Word Problems

Definition System for School Arithmetic

A detailed description of this definition system is beyond the scope of this article. However, these brief descriptions of several features may help to illustrate it. Note that the system *underlies* school arithmetic. Making a teachable school arithmetic with such an underlying system was an accomplishment that occurred decades after the construction of the system.

Language. The definitions are written with words, numbers, and arithmetic symbols (+, −, ×, ÷, =). No mathematical symbols occur beyond those of arithmetic, i.e., the letters used in algebra do not occur.

Definitions. Section 1 of the system consists of a group of "general definitions." These play a role similar to that of the definitions in Book I of the *Elements*: providing a foundation on which the rest of the system is built. Later sections, e.g., "Addition and Subtraction with Whole Numbers," "Multiplication and Division with Whole Numbers," "Addition and Subtraction with

Fractions," each give the few "specific definitions" needed to discuss the subject of the section. These "specific definitions" are derived from the specific definitions in previous sections or from the general definitions in Section 1. For example, the specific definitions in "Addition and Subtraction with Fractions" may draw on general definitions, as well as those in the earlier section "Addition and Subtraction with Whole Numbers." The role of the "specific definitions" is similar to the role of the definitions at the beginnings of Books II–VII of *The Elements*. All the definitions in the definition system of school arithmetic ultimately rely on the first definition of the system—the definition of "unit."

Quantitative relationships. The definition system provides explanations for the algorithms of the four operations on whole numbers and fractions. It also reveals the basic quantitative relationships used to analyze and represent more complicated relationships in arithmetic problems. For example, the two basic quantitative relationships defined in the system are "the sum of two numbers" (with it the operations of addition and subtraction are defined) and "the product of two numbers" (with it the operations of multiplication and division are defined). With these two basic quantitative relationships, other quantitative relationships such as "the sum of two differences," "the product of two sums," "the difference between a sum and a product," and more complicated relationships can be analyzed and represented.

Recreational Word Problems with Variants

Types of word problems originating from ancient civilizations were introduced during the construction of school arithmetic.[i] These problems were used with the approach of juxtaposing variants.[ii] Inclusion of these problem types with their variants made at least two important contributions to school arithmetic. First, while the definition system established by following the approach of the *Elements* emphasizes rigorous reasoning, use of these problem types can inspire students to think in a flexible way—to recognize different variants of a prototypical problem and solve them. Second, the word problems included in school arithmetic's predecessor, commercial arithmetic, were mainly problems aimed at a specific commercial use. The ancient problem types, however, were mainly "recreational mathematics," problems for mental recreation without a practical purpose. Working with these seemingly "useless" problems, in fact, is a significant, even essential part of our mathematical mental activity. Exposing students to this kind of experience obviously broadens their perspective on the nature of mathematics.

References

Swetz, F. (1987). *Capitalism and arithmetic: The new math of the 15th century*. La Salle, IL: Open Court.

Shen, K, Crossley, J. & Lun, A. (1999). *The nine chapters on the mathematical art: Companion & commentary.* New York: Oxford University Press.

[i] For instance, "pursuit problems" are one such type. The *Treviso Arithmetic* (published in 1478), "the earliest known dated, printed arithmetic book" (Swetz, 1987, p. xv) gives the following example:

> A hare is 150 paces ahead of a hound, which pursues him. The hare covers 6 paces, while the hound covers 10. Required is to know how many paces the hound has made when he overtakes the hare. (p. 160)

In the *Nine Chapters*, an ancient Chinese mathematics book, we see similar problems such as:

Now a good walker covers 100 *bu*, while a poor walker covers 60 *bu*. Assume the latter goes 100 *bu* ahead of the former, who catches up with him. Tell in how many *bu* will the two come abreast? (Shen et al., 1999, p. 329)

Such "pursuit problems" were introduced in school arithmetic using contexts like a train "chasing" a car or a bicycle rider "chasing" a walker.

There is an example of another type, a "work problem," in the *Treviso Arithmetic*:

A carpenter has undertaken to build a house in 20 days. He takes on another man and says: "If we build the house together, we can accomplish the work in 8 days." Required is to know how long it would take this other man to build it alone. (Swetz, 1987, p. 163)

Swetz comments that this problem "is a variant of an even more ancient and traditional problem, that of pipes filling a cistern or fountain, common sights in the Mediterranean world" (p. 274). The *Nine Chapters* has an example of a "cistern problem":

Now given a pond which is filled through five canals. Open the first canal and the pond fills in 1/3 day; with the second, it fills in a day; with the third, in 2 ½ days; with the fourth, in 3 days, with the fifth in 5 days. Assume all of them are opened. Tell: how many days are required to fill the pond? (Shen et al., p. 343)

In Chinese elementary mathematics, one can see work problems juxtaposed with cistern problems.

[ii] The endnote above illustrates one kind of variation: change in problem type, in this case, work problem to cistern problem. There are other two common types of problem juxtapositions. One is "same topic, different problem." The famous "I found a stone" series of 22 problems on tablet #4652 of the Yale Babylonian Collection illustrates this:

a) I found a stone. I did not weigh it. A seventh I added. An eleventh I added. I weighed it. 1 *mana*. What was the original weight of the stone?

b) I found a stone. I did not weigh it. A seventh I took away. A thirteenth I took away. I weighed it. 1 *mana*. What was the original weight of the stone?

c) I found a stone. I did not weigh it. A seventh I took away. An eleventh I added. A thirteenth I took away. I weighed it. 1 *mana*. What was the original weight of the stone? . . .

Another juxtaposition is "same type, different topic." The pursuit problem in the endnote above together with the two problems from the *Nine Chapters* below illustrate this:

Now a hare runs 100 *bu* ahead. A dog pursuing at 250 *bu* is 30 *bu* short. Tell: in how many more *bu* will the dog catch up with the hare? . . .

Now given a guest on horseback rides 300 *li* a day. The guest leaves his clothes behind. The host discovers them after 1/3 day, and he starts out with the clothes. As soon as he catches up with the guest, the host gives back the clothes and returns home in ¾ day. Assume the host rides without a stop. Tell: how far can he go in a day? (Shen et al., 1999, p. 330)

These three kinds of juxtapositions were all introduced as school arithmetic was constructed.